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Lecture Notes on Quantum Mechanics No. 11
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Probabilities for Quantum Histories II. Multitime Histories

References:

CQT = R. B. Griffiths, *Consistent Quantum Theory* (Cambridge, 2002)

Contents

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1 Introduction

• The Born rule only applies to closed quantum systems, and only gives (conditional) probabilities in the case of a sample space with two-time histories. Extending it to histories of a closed quantum system involving three or more times is not trivial because of the phenomenon of quantum interference, and was first carried out in the 1980's and required some new ideas.

Born rule does not extend to all history sample spaces; consistency conditions

⊙ In particular, there is no (known) way of extending the Born rule to *all* sample spaces of histories without running into difficulties and inconsistencies. In this respect the situation is not unlike that encountered when one tries to extend logical operations which make perfectly good sense for a classical phase space to logical operations on quantum properties, subspaces of the Hilbert space. Progress has come about by restricting the class of "objects" one wants to talk about, and refusing to combine cases in which projectors do not commute, on the grounds that they are meaningless: quantum mechanics, at least as understood at present, can assign them no meanings. In a similar way it turns out that one can extend the Born rule only to certain history sample spaces which satisfy what are called *consistency conditions*. If these are not satisfied the sample spaces represent mutually-exclusive histories, but even if they occur inside a closed system we do not know how to assign them probabilities starting with Schrödinger's equation.

• The Born rule assigns probabilities using unitary time development, i.e., Schrödinger's equation, as a tool. For histories of involving three or more times, unitary time development is needed to test for consistency as well as assign probabilities.

⊙ The material in these notes is basically the same as in CQT Chs. 10 and 11. However, they begin in Sec. 2 with the special case discussed in Sec. 11.6 of CQT, which is easier to understand than the general case. The latter is the subject of Sec. 3

• Rules are best understood by applying them to numerous examples. There are a large number of (fairly) simple examples in CQT Chs. 12 and 13, so these notes have only a few.

2 Initial Pure State

◦ Reference: CQT Sec. 11.6

2.1 Chain kets and consistency

⊙ Simplest situation: histories all begin with pure state $|\psi_0\rangle$, so are of the form

$$Y^\alpha = [\psi_0] \odot P_1^{\alpha_1} \odot P_2^{\alpha_2} \odot \cdots P_f^{\alpha_f} = [\psi_0] \odot X^\alpha, \tag{1}$$

We now think of α as the string

$$\alpha = (\alpha_1, \alpha_2, \ldots \alpha_f), \tag{2}$$

and suppose (for simplicity; this is not absolutely necessary) that the histories X^α are drawn from the product history sample space constructed using the decompositions

$$I_m = \sum_{\alpha_m} P_m^{\alpha_m} \tag{3}$$

⊙ For each history α define the corresponding *chain ket*

$$|\alpha\rangle = P_f^{\alpha_f} T(t_f, t_{f-1}) \cdots P_2^{\alpha_2} T(t_2, t_1) P_1^{\alpha_1} T(t_1, t_0) |\psi_0\rangle. \tag{4}$$

◦ Notice that the chain ket $|\alpha\rangle$ is an element of the one-time Hilbert space $\mathcal{H}$ and not the histories Hilbert space $\breve{\mathcal{H}}$. The α is a label. Also $|\alpha\rangle$ is (in general) not normalized even if, as we shall suppose, $|\psi_0\rangle$ is normalized.

⊙ The *consistency condition* for the family (1) is

$$\langle\alpha|\beta\rangle = 0 \text{ for } \alpha \neq \beta \tag{5}$$

Where $\beta = (\beta_1, \beta_2, \ldots \beta_f)$ labels a different history from the same sample space of histories; $\alpha \neq \beta$ means that there is at least one j in the interval from 1 to f such that $\alpha_j \neq \beta_j$. Thus the consistency condition states that the inner product of the chain kets $|\alpha\rangle$ and $|\beta\rangle$ must vanish whenever α and β are *distinct* histories in the sample space.

⊙ *If the consistency conditions in* (5) *are satisfied*, one assigns (conditional) probabilities

$$\Pr(\alpha \,|\, \psi_0) = \Pr(\alpha) = \langle\alpha|\alpha\rangle \tag{6}$$

to the different histories in the sample space. These in turn generate the probabilities for the different (history) projectors in the corresponding event algebra in the usual way.

• Since the history projectors in (1) sum to $[\psi_0] \odot I \odot \cdots$, we need another one, $(I - [\psi_0]) \odot I \odot \cdots$ to complete the history identity $\breve{I}$. To this additional history we assign probability 0.

2.2 Examples

• In order to understand the significance of the consistency conditions (5) and the probabilities (6) that result when they are satisfied, we need to look at a number of special cases and examples.

⊙ First example: $f = 1$, the histories only involve two times t_0 and t_1. In this case the consistency condition is automatically satisfied, since the strings α and β are one label long, and $\alpha \neq \beta$ is thus the same as $\alpha_1 \neq \beta_1$. Since they are drawn from the same decomposition of the identity, $P_1^{\alpha_1}$ and $P_1^{\beta_1}$ are automatically orthogonal for $\alpha_1 \neq \beta_1$, and therefore the consistency condition (5) is satisfied.

$\odot$ Even for $f > 1$ it is always the case that if $\alpha_f \neq \beta_f$, then $\langle\alpha|\beta\rangle = 0$.

$\star$ Exercise. Show this.

• Consequently, checking consistency requires looking at cases in which $\alpha_f = \beta_f$, and checking that $\langle\alpha|\beta\rangle = 0$ when some of the earlier elements in the labels are unequal.

$\odot$ The Born rule applies to two-time histories, and therefore when using the Born rule one can ignore consistency conditions.

• This is true even for the general form of the Born rule discussed in a previous chapter of these notes, though that lies outside the present discussion, limited to histories of the form (1) with a fixed initial (pure) state.

$\odot$ Second example. Let $f = 2$ and suppose we are dealing with a spin-half particle. For simplicity let $T(t,t') = I$, thus no magnetic field is present, and suppose that $|\psi_0\rangle = |z^+\rangle = |0\rangle$. Let v and w be two arbitrary directions in space, or points on the Bloch sphere, and use decompositions

$$I_1 = [v^+] + [v^-]; \quad I_2 = [w^+] + [w^-]. \tag{7}$$

at times t_1 and t_2, respectively.

• There are then four histories in the sample space. Use the notation $|(v^+, w^-)\rangle = |(+,-)\rangle$, etc., for the corresponding chain kets. Are they mutually orthogonal as required by the consistency conditions? In light of the automatic orthogonality whenever $\alpha_f \neq \beta_f$, see above, we need only check cases in which the final (w) label is the same, but the first (v) label is different, for example,

$$\langle(v^+, w^+)|(v^-, w^+)\rangle = \langle z^+|v^+\rangle\langle v^+|w^+\rangle\langle w^+|v^-\rangle\langle v^-|z^+\rangle. \tag{8}$$

$\star$ Exercise. Work out the right side starting with the definition of the chain kets in (4)

• In order for (8) to vanish, at least one of the factors on the right side must be zero. How can this be achieved? There two distinct possibilities. If $v = z$ (or $-z$, which for our purposes amounts to the same thing), then either $\langle z^+|v^+\rangle$ or $\langle v^-|z^+\rangle$ will be zero and the right side of (8) will be zero. Otherwise both $\langle z^+|v^+\rangle$ and $\langle v^-|z^+\rangle$ will be nonzero. The other possibility is that $v = w$ (or $-w$), in which case either $\langle v^+|w^+\rangle$ or $\langle w^+|v^-\rangle$; otherwise both are nonzero.

• Next one needs to check the case where w^+ in (8) is replaced with w^-, but the conclusion is exactly the same conclusion. A quick way to see this is to replace $|w^+\rangle\langle w^+|$ in the middle of the right side of (8) with $I - |w^-\rangle\langle w^-|$.

$\star$ Exercise. Explain why this works.

$\odot$ Thus consistency is violated unless the intermediate time v basis is (i) the same as the z basis, or (ii) the same as the w basis. In the special case in which w is z (or $-z$) the only possibility is that v is also z (or $-z$). In any case, consistency is a very restrictive condition.

$\star$ Exercise. What are possible choices of v and w (i.e., bases at times t_1 and t_2 given an initial state $|\psi_0\rangle = |z^+\rangle$, but instead of assuming that $T(t,t') = 1$ we assume that in the standard basis

$$T(t_1,t_0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad T(t_2,t_1) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \tag{9}$$

[Hint. In geometrical terms $T(t_1,t_0)$ is a 180° rotation that interchanges the $+x$ and $+z$ axes of the Bloch sphere, whereas $T(t_2,t_1)$ represents a rotation of $+\pi/2$ about the z axis.]

$\star$ Exercise. Show that in the case of spin-half, consistency will be violated if there are more than two nonzero chain kets. How does this generalize to the case where the Hilbert space is of dimension $d > 2$?

• For other (fairly) simple examples of consistent (and inconsistent) families, see Chs. 12 and 13 of CQT.

3 General Consistent Families

• In this section we consider the case of a general sample space of product histories: each history is of the form

$$Y = F_0 \odot F_1 \odot \cdots F_f, \tag{10}$$

with each F_j a projector indicating a particular property at time t_j. However, the sample space need not be a product sample space in which all the F_j at time t_j are drawn from a single decomposition of I.

• While it is convenient to assume that only a finite number $f+1$ of times are involved, it is not necessary that each element of the sample space have a nontrivial projector at each of these times. At a particular time t_j the corresponding F_j might be the identity operator, the property that is always and trivially true. Or, given a history defined with projectors at a given set of times, one can always introduce additional times and insert the identity at these times; this new history projector has the same physical significance as the original one.

⊙ In a closed or isolated quantum system with well-defined unitary time-development operators $T(t,t')$ one can associated with every product history of the form (10) a *chain operator*

$$K(Y) = F_f T(t_f, t_{f-1}) F_{f-1} \cdots T(t_2, t_1) F_1 T(t_1, t_0) F_0 \tag{11}$$

which generalizes the the notion of a chain ket introduced earlier.

• Note that $K(Y)$ is an operator on the one-time Hilbert space $\mathcal{H}$, not an operator on the history space $\breve{\mathcal{H}}$, so one can think of $Y \to K(Y)$ as a linear map from operators on the "big" histories Hilbert space $\breve{\mathcal{H}}$ to operators on the "small" Hilbert space $\mathcal{H}$. Linearity means that

$$K(Y' + Y'' + \cdots) = K(Y') + K(Y'') + \cdots . \tag{12}$$

⊙ Next define the Frobenius (or Hilbert-Schmidt) inner product of operators acting on $\mathcal{H}$: inner product

$$\langle A, B \rangle = \mathrm{Tr}[A^\dagger B]. \tag{13}$$

This has the usual properties of an inner product: $\langle A, B \rangle = \langle B, A \rangle^*$; antilinear in the first argument and linear in the second; $\langle A, A \rangle \geq 0$, with equality if and only if $A = 0$ is the zero operator.

• The general *consistency condition* for a sample space $\{Y^\alpha\}$ of product history projectors, i.e., a decomposition of $\breve{I}$ in which each projector is of the form (10), takes the form

$$\langle K(Y^\beta), K(Y^\gamma) \rangle = 0 \text{ for } \beta \neq \gamma. \tag{14}$$

• Here β and γ come from the set of labels $\{\alpha\}$ used to label the histories in the decomposition $\{Y^\alpha\}$.

∘ In the case of a product sample space it is natural (but not essential) to use strings as labels: $\alpha = (\alpha_0, \ldots \alpha_f)$. However, the present discussion is not limited to product sample spaces, though we do assume that all histories are product histories of the general form (10), as this was used in defining the chain operators (11).

• A sample space for which (14) holds is called a *consistent* sample space, and the corresponding event algebra is referred to as a *consistent family*. However, "consistent family" can also refer to the sample space; this does not cause confusion because the event algebra $\mathcal{E}$ is generated by the sample space $\{Y^\alpha\}$ in the sense that it contains all projectors of the form

$$Y = \sum_\alpha \pi_\alpha Y^\alpha, \tag{15}$$

where each π_α can be either 0 or 1.

$\odot$ Given a consistent family (consistent sample space) the *weight* $W(Y)$ of any history Y in the family, i.e., any Y of the form (15) is given by by the formula:

$$W(Y) = \mathrm{Tr}[K^\dagger(Y)K(Y)]. \tag{16}$$

• Because $K(Y) = \sum_\alpha \pi_\alpha Y^\alpha$ and because we are assuming that the consistency conditions (14) are satisfied, the weight of any history of the form (15) is given by

$$W(Y) = W(\sum_\alpha \pi_a l Y^\alpha) = \sum_\alpha \pi_a W(Y^\alpha), \tag{17}$$

i.e., the weights on the histories in the sample space determine the weights of histories in the event algebra in the same manner as probabilities in an event algebra are determined by probabilities of elements in the sample space.

• We refer to the $W(Y)$ as *weights* rather than probabilities because there is no requirement that they add up to 1. They function in much the same way as stochastic matrices, or products of such matrices, in a Markov process: they are used to generate a probability distribution given some additional constraints or assumptions (such as an initial state or an initial probability distribution), as will become clear from considering various examples.

◦ The situation is similar to that which arises for the general Born rule in which one allows a general decomposition of the identity $\{P^j\}$ at an initial time and a different decomposition $\{Q^k\}$ at a later time.

⋆ Exercise. Work out the chain operators for the case of a pure initial state considered above in Sec. 2, and relate the results on consistency and weights given here to those in Sec. 2.