33-445 Advanced Quantum Physics Fall Semester, 2012
Assignment No. 13. Due Friday, December 7

COURSE WEB PAGE:
http://quantum.phys.cmu.edu/quad/

NOTICE. To facilitate studying for the final exam, solutions to this problem assignment will be handed out in class on Friday morning at the same time the assignments are turned in for grading. For this reason, no late assignments will be accepted: if you want yours to be graded get it in on time.

The main topic for this week's classes is coherent states for a harmonic oscillator. If some time is left over it will be used for a course review.

READING: Sources

Townsend = *A Modern Approach to Quantum Mechanics*, 2d ed

HO = Harmonic Oscillator (Course web page). This includes and replaces the earlier Harmonic Oscillator Notation.

READING: Topics

Position space wave functions: Townsend, Secs. 7.4 and 7.9; HO Sec. 3. We will only be using the approach in Townsend's Sec. 7.4, and you are not responsible for material in Sec. 7.9.

Inversion symmetry: Townsend, Sec. 7.10

Coherent states: Townsend, Sec. 7.8; HO Sec. 4.

EXERCISES:

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course.

2. a) Starting with the lowering operator a expressed in terms of the dimensionless position $\bar{X}$ and momentum $\bar{P}$ operators show that it can be written using the dimensionless momentum variable $v = p/\eta$, $\eta = \sqrt{\hbar m \omega}$, in the form

$$a = \frac{i}{\sqrt{2}}\Big(v + \frac{\partial}{\partial v}\Big).$$

What is the corresponding expression for $a^\dagger$? [Hint. It may be useful to look at Townsend Problem 6.2.]

b) Turn the equation $a|n = 0\rangle = 0$ into a differential equation in v for the ground state $\hat{\phi}_0(v) = \langle v|0\rangle$ of the harmonic oscillator in the momentum representation. Solve this equation and choose the coefficient such that the ground states in momentum and position are related by the (dimensionless) Fourier transform:

$$\hat{\phi}_0(v) = \int_{-\infty}^{\infty} \langle v|u\rangle \phi_0(u)\, du, \quad \langle v|u\rangle = e^{-iuv}/\sqrt{2\pi}; \quad \phi_0(u) = e^{-u^2/2}/\pi^{1/4}.$$

c) Use $\hat{\phi}_1(v) = \langle v|1\rangle = \langle v|a^\dagger|0\rangle$, with $a^\dagger$ the differential operator you found in (a) and $\hat{\phi}_0(v)$ the function determined in (b), to calculate $\hat{\phi}_1(v)$, the wave function for the first excited state of the oscillator in the momentum representation. Note: Once the phase of $\hat{\phi}_0(v) = \langle v|0\rangle$ has been specified, as in (b) above, that of $\hat{\phi}_1(v) = \langle v|1\rangle$ is fixed by the fact that $|1\rangle = a^\dagger|0\rangle$.

d) Show that $|n\rangle = (a^\dagger)^n|0\rangle/\sqrt{n!}$ implies that the (dimensionless) momentum and position wave functions are related by

$$\hat{\phi}_n(v) = (-i)^n \phi_n(u = v).$$

3. Townsend Problem 7.14. In addition answer the same question assuming that the particle is initially in the *first excited state* of the pendulum of length L: what is the probability that the particle will be (found) in the ground state of the new pendulum of length $4L$?

4. Find an expression for the inner product $\langle\alpha|\beta\rangle$ of two coherent states $|\alpha\rangle$ and $|\beta\rangle$. Show that $|\langle\alpha|\beta\rangle|$ depends only on $|\alpha - \beta|$ by expressing the former as an explicit function of the latter.

5. Find $\langle\alpha|\bar{X}\bar{P} + \bar{P}\bar{X}|\alpha\rangle$ for the coherent state $|\alpha\rangle$. For which values does it vanish? Does this seem plausible in view of what you would expect for a classical harmonic oscillator? [Hint: First write the operator in terms of a and $a^\dagger$.]

6. a) Find an expression for the wave packet in (dimensionless) momentum space $\hat{\phi}(\alpha; v)$ by writing the displacement operator $D(\alpha)$ in the form

$$D(\alpha) = e^{\cdots} e^{\cdots \bar{P}} e^{\cdots \bar{X}},$$

where the $\cdots$ are different things to be filled in (functions of α). Next use $\bar{X}$ in the form $i\partial/\partial v$ to transform $D(\alpha)|0\rangle$ into a displaced-in-momentum-space wave packet (thus a function of v) multiplied by a v-dependent phase. (You can obtain the same result by applying a Fourier transform to the position space wave packet $\phi(\alpha; u)$ given in the notes, but using the technique involving $D(\alpha)$ written as a product of exponentials is useful in understanding what to do with latter.)

b) Discuss the physical significance of the resulting wave packet $\hat{\phi}(\alpha; v)$ as a function of time, i.e., as α moves in a circle in the complex plane. What can you say about (approximate) momentum and position as a function of time, based on this wave packet? Assume $|\alpha|$ is large if that makes the discussion easier.