33-445 Advanced Quantum Physics Fall Semester, 2012
Assignment No. 12. Final Version (corrected Nov. 28). Due Friday, November 30

COURSE WEB PAGE:
http://quantum.phys.cmu.edu/quad/

READING: Sources
Townsend = *A Modern Approach to Quantum Mechanics*, 2d ed
HON = Harmonic Oscillator Notation (Course web page)

READING: Topics
Scattering: Townsend, Sec. 6.10
Tunneling: Townsend, Sec. 6.10
Harmonic oscillator using raising and lowering operators: Townsend, Secs. 7.1 to 7.4. Lectures will use a slightly different notation, see HON.
Harmonic oscillator: zero-point energy, large n limit, time dependence: Townsend, Secs. 7.5 to 7.7

READING AHEAD:
Coherent states: Townsend, Sec. 7.8.
Inversion symmetry: Townsend, Sec. 7.10
Townsend Sec. 7.9, solving the Schrödinger equation in position space, is an alternative approach to obtaining the energy eigenstates in the position representation, something already explored in Sec. 7.4. It is a useful strategy for solving some other differential equations, e.g., the quantum hydrogen atom.

EXERCISES:

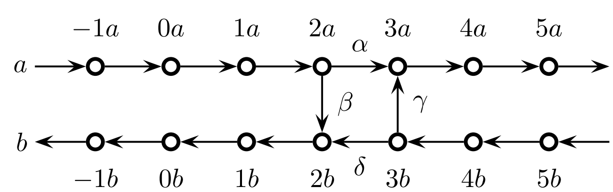
1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course.

2. The following is a toy model of scattering in one dimension. The particle can be on either of two tracks, labeled a and b, with a position given by the integer m. The state $|ma\rangle$ is interpreted as meaning that the particle is at position $x = m$ and moving to the right, while $|mb\rangle$ means $x = m$ and the particle is moving to the left. The time development operator is $T = S$, where the shift S is

$$S|ma\rangle = |(m+1)a\rangle, \quad S|mb\rangle = |(m-1)a\rangle,$$

with the following exceptions, which mimic the effect of a potential barrier:

$$S|2a\rangle = \alpha|3a\rangle + \beta|2b\rangle, \quad S|3b\rangle = \gamma|3a\rangle + \delta|2b\rangle.$$



a) Use unitarity to put conditions on the amplitudes $\alpha, \beta, \gamma, \delta$. What choice should be made for these four amplitudes (the answer is not unique) so that the probability of a particle being transmitted through the barrier from the left to the right is equal to τ, a real number in the range $0 < \tau < 1$? [Hint. A finite unitary matrix can be characterized either by the fact that its rows form an orthonormal system of vectors, or its columns form an orthonormal system; one implies the other.]

b) Show that the transmission coefficient for a particle moving from right to left is also equal to τ.

c) Optional. Show that the two transmission coefficients need *not* be the same if one replaces the quantum toy model with a classical hopping model associated with the same diagram, in which the hopping probabilities are equal to one in the direction of the arrows, except for the four arrows labeled $\alpha, \beta, \gamma, \delta$, where these letters should be interpreted as (real) hopping probabilities satisfying

$$\alpha + \beta = 1, \quad \gamma + \delta = 1.$$

d) Suppose a two-state detector is added on the right side of the barrier, where it detects a particle as it hops from $3a$ to $4a$, i.e., if the detector state is indicated by $|n\rangle$, then $T|3a, n\rangle = |4a, 1-n\rangle$. Find the probability that the detector will have detected the particle at time $t = 4$, given the following initial state at $t = 0$:

$$|\Psi_0\rangle = \Big(|0a\rangle + |0b\rangle\Big) \otimes |n = 0\rangle/\sqrt{2}.$$

e) Given the initial state in (d), and assuming that at $t = 4$ the detector has detected the particle (i.e., $n = 1$), what can you say about the particle's location, the value of m, for $0 < t < 4$? Given an answer which is as precise as possible, consistent with the laws of quantum mechanics (and the simplifications inherent in a toy model).

3. Based upon Townsend Prob. 6.21. Consider reflection from and transmission through a potential energy barrier

$$(2m/\hbar^2)V(x) = (\lambda/b)\delta(x),$$

assuming

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{for } x < 0, \\ Ce^{ikx} & \text{for } x > 0. \end{cases}$$

a) Find the ratios B/A and C/A and check that they have the values you would expect at $\lambda = 0$ and $\lambda = +\infty$. Use them to calculate the transmission T and reflection R coefficients as a function of λ.

b) One would expect to get the same result as in Townsend's (6.144) by using a square potential barrier in which $V_0 \to \infty$ and $a \to 0$ in a suitable manner. Discuss

4. Townsend Prob. 6.25, but replace his final question ("How would you expect ... the applied voltage?") with the following: Suppose the work function is 1 eV. What electric field strength $|\mathbf{E}|$ is needed to produce a transmission coefficient of $T = 10^{-10}$? Convert your answer to volts/nanometer and ask yourself if it seems reasonable. Suppose this electric field is then decreased by 10%. By what factor does T change?

5. The quantities $\xi = \sqrt{\hbar/m\omega}$, $\eta = \sqrt{\hbar m\omega}$, and $\nu = \xi\omega = \sqrt{\hbar\omega/m}$ define a characteristic length, momentum, and velocity for a quantum harmonic oscillator, where m is the mass and ω the angular frequency.

a) Imagine a classical oscillator of mass m, angular frequency ω and total energy $\hbar\omega/2$. How are the properties of its orbit be related to ξ, η, and ν?

b) Find approximate values of ξ and ν for the following cases: (i) A 10 g mass suspended in earth's gravity on a thread of length 10 cm; (ii) A Be ion (atomic mass 9) suspended in a trap where it oscillates at a frequency ($\omega/2\pi$) of 10 MHz; (iii) A Weber bar (used in attempts to detect gravitational waves) of mass 1000 kg and a frequency of 1000 Hz.

6. a) For a harmonic oscillator evaluate the following, where $|n\rangle$ and $\langle n|$ refer to number states:

$$\langle 3|\,(a^\dagger)^2\,|1\rangle, \quad \langle 2|\,a^2(a^\dagger)^2\,|2\rangle.$$

b) Show that $\langle n'|\,a^p(a^\dagger)^q\,|n\rangle = 0$ unless the nonnegative integers n, n', p, q satisfy the condition $n' = n + q - p$, in which case it has a nonzero value which you should calculate.

c) Show that $\langle n'|\,(a^\dagger)^q a^p\,|n\rangle = 0$ unless $n' = n + q - p$, but it is also zero in some other cases. Discuss. Write down an expression for its value when it is not zero.

7. Townsend Problem 7.9. Check that the Ehrenfest conditions, Townsend's (6.33) and (6.34), are actually satisfied.