33-445 Advanced Quantum Physics Fall Semester, 2012
Assignment No. 9. Not to be turned in

ANNOUNCEMENT: There will be an hour exam during class the morning of Friday, Nov. 2. It will be closed book, closed notes, no pocket calculators. Bring a sharp pencil.

The examination will be available at 9:15 a.m., and must be completed by 10:30 a.m.

The examination will be on the material covered in class through Wednesday, Oct. 24, and found in assignments 1 to 8, with particular emphasis on the material covered since the last hour exam, Assignments 6-8. You are *not* responsible on this examination for topics in quantum measurements, a subject begun in class on Oct. 24.

COURSE WEB PAGE:
http://quantum.phys.cmu.edu/quad/

READING: Sources
Townsend = *A Modern Approach to Quantum Mechanics*, 2d ed
CQT = Consistent Quantum Theory. Individual chapters at: http://quantum.phys.cmu.edu/CQT/
QMM = Quantum Measurements (Course web page)

READING: Topics
Partial trace: CQT Sec. 6.5
Reduced density operator: CQT Secs. 15.3 to 15.5.
Ideal quantum measurements: CQT Secs. 17.1, 17.2, 17.5, 18.1, 18.2; QMM Sec. 3.
Partial quantum measurements: CQT Secs. 18.1, 18.2; QMM Sec. 4.
Einstein-Podolsky-Rosen-Bell: Townsend Secs. 5.4, 5.5; CQT Ch. 24 (Ch. 23 is relevant, but rather long and tedious); QMM Sec. 4.4.

READING AHEAD:
Wave mechanics in one dimension: Townsend Secs. 6.1 to 6.5

EXERCISES:

1. Let $U(t) = T(t,0)$ be the unitary time development operator for a system with time-independent Hamiltonian.

a) What is $U(t=0)$?

b) Express $U(-t)$ in terms of $U(t)$.

c) Express $U(t_1 - t_2)$ in terms of $U(t_1)$ and $U(t_2)$; provide some explanation.

d) Suppose that for a 3-dimensional Hilbert space it is the case that

$$U(t) = \begin{pmatrix} e^{-2i\omega t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\omega t} \end{pmatrix},$$

where ω is a quantity with dimensions of inverse time. Find the Hamiltonian H as a matrix, and provide some justification for your answer.

2. Consider an entangled state

$$|\Psi\rangle = \sqrt{1/3}\,|z^+\rangle_a \otimes |x^+\rangle_b + \sqrt{2/3}\,|z^-\rangle_a \otimes |z^-\rangle_b$$

on the tensor product $\mathcal{H}_a \otimes \mathcal{H}_b$ of two spin-half particles. (Note that the two b kets used in this expression are not orthogonal to each other.)

a) Apply $[z^+]_a = [z^+]_a \otimes I_b$ to $|\Psi\rangle$ and use the result to find the probability $\Pr([z^+]_a)$ as given by the Born rule. Use the same procedure with $[z^-]_a$ in place of $[z^+]_a$ to find $\Pr([z^-]_a)$.

b) Find the reduced density operator $\rho_a = \mathrm{Tr}_b(|\Psi\rangle\langle\Psi|)$ in the z basis: either write your answer in terms of dyads involving $|z^+\rangle$, $|z^-\rangle$, or else as a 2×2 matrix.

c) Evaluate the probabilities $\Pr([z^+]_a)$ and $\Pr([z^-]_a)$ by taking appropriate traces using the density operator ρ_a you found in (b).

d) Use the density operator in (b) to find $\Pr([x^-]_a)$ using the fact that

$$[x^-] = \frac{1}{2}\big(|z^+\rangle\langle z^+| - |z^+\rangle\langle z^-| - |z^-\rangle\langle z^+| + |z^-\rangle\langle z^-|\big) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

e) Check your answer to (d) by evaluating $|\Phi\rangle = [x^-]_a|\Psi\rangle$ and then calculating $\|\Phi\|^2 = \langle\Phi|\Phi\rangle$.

3. a) Show that the following operator is a projector:

$$P = \begin{pmatrix} 1/2 & i/2 & 0 \\ -i/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

b) Suppose that V is a (time-independent) Hermitian operator and $|\psi(t)\rangle$ a solution to Schrödinger's equation,

$$V = \sum_j v_j P^j = v_1 \begin{pmatrix} 1/2 & i/2 & 0 \\ -i/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + v_2 \begin{pmatrix} 1/2 & -i/2 & 0 \\ i/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + v_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad |\psi(t)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\omega t \\ -i\sin\omega t \\ 1 \end{pmatrix},$$

where v_1, v_2 and v_3 are the eigenvalues of V, and they are all different. Find the three probabilities $\Pr(V = v_1)$, $\Pr(V = v_2)$ and $\Pr(V = v_3)$ at the time t.

c) Find a Hamiltonian H as a 3×3 matrix such that for this Hamiltonian $|\psi(t)\rangle$ in (b) is a solution of Schrödinger's equation.

d) One of the probabilities you calculated in (b) is independent of time. Show that the corresponding project P^j commutes with the Hamiltonian H that you found in (c). Then give a general argument to show that if some projector P commutes with H, the corresponding probability will not depend upon time.

4. Consider two entangled kets

$$|\Psi\rangle = \sum_{jp} M_{jp}|a^j\rangle \otimes |b^p\rangle, \quad |\Phi\rangle = \sum_{kq} N_{kq}|a^k\rangle \otimes |b^q\rangle$$

on the tensor product $\mathcal{H}_a \otimes \mathcal{H}_b$, where $\{|a^j\rangle\}$ and $\{|b^p\rangle\}$ are orthonormal bases of $\mathcal{H}_a$ and $\mathcal{H}_b$.

a) Write $|\Psi\rangle\langle\Phi|$ as a sum of dyads on $\mathcal{H}_a$ tensored with dyads on $\mathcal{H}_b$.

b) Consider the partial traces

$$R = \mathrm{Tr}_b\big(|\Psi\rangle\langle\Phi|\big) = \sum_{jk} R_{jk}|a^j\rangle\langle a^k|, \quad S = \mathrm{Tr}_a\big(|\Psi\rangle\langle\Phi|\big) = \sum_{pq} S_{pq}|b^p\rangle\langle b^q|.$$

Find expressions for the matrices R and S in terms of the matrices M and N. Can you write R and S as matrix products in a simple way?

5. Let $\{P^1, P^2, P^3\}$ and $\{Q^1, Q^2, Q^3\}$ be two distinct decompositions of the identity, and $|\psi_0\rangle$ a normalized state.

a) Consider a family of histories $\{Y^\alpha\}$ for a set of times $t_0 < t_1 < t_2$ with initial state $[\psi_0] = |\psi_0\rangle\langle\psi_0|$, where

$$Y^1 = [\psi_0] \odot P^1 \odot (Q^1 + Q^2),$$
$$Y^2 = [\psi_0] \odot (P^2 + P^3) \odot Q^3$$

Find two additional histories Y^0, Y^3 and Y^4 such that

$$\sum_{\alpha=1}^{4} Y^\alpha = [\psi_0] \odot I \odot I, \quad \sum_{\alpha=0}^{4} Y^\alpha = I \odot I \odot I.$$

b) State the conditions that need to be satisfied by the chain kets $\{|Y^\alpha\rangle\}$ for $1 \le \alpha \le 4$, where

$$|Y^1\rangle = (Q^1 + Q^2)T(t_2, t_1)P^1T(t_1, t_0)|\psi_0\rangle,$$

and the other $|Y^\alpha\rangle$ for $\alpha = 2, 3, 4$ are defined in a similar way, so that probabilities can be consistently assigned to these histories (conditional on the initial $[\psi_0]$). What are the corresponding probabilities, again in terms of the chian kets, when these conditions are satisfied?

c) What is $\Pr(Q^3 \text{ at } t_2 \mid [\psi_0] \text{ at } t_0)$? Express your answer in terms of chain kets (as in (b)).