33-445 Advanced Quantum Physics Fall Semester, 2012
Assignment No. 8 (FINAL VERSION) Due Friday, October 26

ANNOUNCEMENT: There will be an hour exam during class the morning of Friday, Nov. 2. It will be closed book, closed notes, no pocket calculators. Bring a sharp pencil. The examination will be on the material covered in class through Wednesday, Oct. 24, and found in assignments 1 to 8, with particular emphasis on the material covered since the last hour exam, Assignments 6-8. More details later.

COURSE WEB PAGE:
http://quantum.phys.cmu.edu/quad/

READING: Sources
Townsend = *A Modern Approach to Quantum Mechanics*, 2d ed
CQT = Consistent Quantum Theory. Individual chapters at: http://quantum.phys.cmu.edu/CQT/

The following items are on the course web page. They were prepared for a previous course, so there will be some overlap with other notes, and they contain some things we will not be covering.
UTDBR = "Unitary Time Development and Born Rule"
PRQHI = "Probabilities for Quantum Histories I"
PRQHII = "Probabilities for Quantum Histories II" Secs. 1,2.

READING: Topics
Toy models: CQT Secs. 2.5, 6.3, 7.4, 12.1, 12.2; UTDBR
Consistency conditions and probabilities of histories: CQT Chs. 10 and 11; in particular Sec. 11.6. We will not be considering the most general case discussed in Ch. 10 (summarized in Sec. 3 of PRQHII), and will instead focus on chain kets, Sec. 11.6; PRQHII Secs 1, 2.

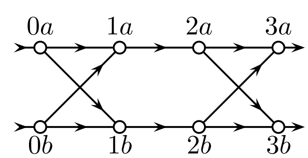
Interference: CQT Ch. 13. Feynman *Lectures on Physics*, Vol. III, Ch. 1; also in Vol. I. Ch. 37
Density operators: Townsend Secs. 5.7, 5.8. CQT Secs. 15.1 to 15.4.

READING AHEAD:
Measurements: CQT Secs. 17.1, 17.2, 17.5, 18.1, 18.2.

EXERCISES:

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course.

2. This problem is closely connected with the handout *Toy Mach-Zehnder with Measurements* on the course web page. The unitary time transformation in the absence of detectors is given by $T = S$ with



$$S|0a\rangle = (1/\sqrt{2})|1a\rangle + (1/\sqrt{2})|1b\rangle, \quad S|1a\rangle = e^{i\phi_a}|2a\rangle, \quad S|2a\rangle = (1/\sqrt{2})|3a\rangle + (1/\sqrt{2})|3b\rangle$$
$$S|0b\rangle = -(1/\sqrt{2})|1a\rangle + (1/\sqrt{2})|1b\rangle, \quad S|1b\rangle = e^{i\phi_b}|2b\rangle, \quad S|2b\rangle = -(1/\sqrt{2})|3a\rangle + (1/\sqrt{2})|3b\rangle.$$

a) Assume an initial state $|\psi_0\rangle = |0b\rangle$ (note that this is different from the $|\psi_0\rangle$ in the handout), and work out $|\psi_t\rangle = T^t|\psi_0\rangle$ for $t = 1$, 2, and 3. Apply the Born rule at time $t = 3$ to find the probabilities that the particle will be at $3a$ or $3b$. Insofar as possible express your answers in terms of $\Delta := \phi_a - \phi_b$. What choice of phases ϕ_a and ϕ_b leads to the maximum probability at $3a$? At $3b$? How do your results provide evidence that the particle was (in some sense) in both the $1a \to 2a$ and $1b \to 2b$ arms of the interferometer between $t = 1$ and $t = 2$?

b) Consider the family with support

$$[0b] \odot \{[1a], [1b]\} \odot I \odot \{[3a], [3b]\}.$$

Discuss its consistency for different values of ϕ_a and ϕ_b. Is it posible to replace the pair $\{[1a], [1b]\}$ by a different pair of pure state projectors $\{[\chi^1], [\chi^2]\}$, with $[\chi^1] + [\chi^2] = [1a] + [1b]$, which is consistent for all values of ϕ_a and ϕ_b?

c) Now add a measuring device with states $|0\hat{a}\rangle$ and $|1\hat{a}\rangle$ which detects the particle as it hops from $1a$ to $2a$, by setting $T = SR$, R the identity except for $R|1a, p\hat{a}\rangle = |1a, (1 - p)\hat{a}\rangle$, where $p = 0$ and 1 are the ready and triggered states of the detector, and the initial state is $|\Psi_0\rangle = |\psi_0\rangle \otimes |0\hat{a}\rangle = |0b, 0\hat{a}\rangle$. With this new arrangement work out $|\Psi_t\rangle = T^t|\Psi_0\rangle$ for $t = 1$, 2, and 3, and the probability that the particle will be at $3a$ or $3b$ at $t = 3$. How do the phases ϕ_a and ϕ_b influence these probabilities? Also discuss the consistency of the family $[0b, 0\hat{a}] \odot \{[1a], [1b]\} \odot I \odot \{[3a], [3b]\}$, corresponding to that in (b), given the new arrangement with the detector present.

d) Feynman claims that in the case of a weak light source if one segregates the data on where the electron (the interfering particle) arrives, depending on whether or not the hole it passes through was or was not detected, there will be an interference pattern for those cases in which the path was *not* detected, but no interference for the case in which it *was* detected. This can be studied using the toy Mach-Zehnder by adding two detectors, so the Hilbert space is now $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_{\hat{a}} \otimes \mathcal{H}_{\hat{b}}$, the initial state is $|\Psi_0\rangle = |0b, 0\hat{a}, 0\hat{b}\rangle$, thus both detectors are ready at the initial time, and $T = SR$, with R the identity except for

$$R|1a, 0\hat{a}, q\hat{b}\rangle = \alpha|1a, 0\hat{a}, q\hat{b}\rangle + \beta|1a, 1\hat{a}, q\hat{b}\rangle, \quad R|1b, p\hat{a}, 0\hat{b}\rangle = \alpha|1b, p\hat{a}, 0\hat{b}\rangle + \beta|1b, p\hat{a}, 1\hat{b}\rangle,$$

where $|\alpha|^2 + |\beta|^2 = 1$. Study this model by first working out $|\Psi_t\rangle = T^t|\Psi_0\rangle$ for $t = 1$, 2, and 3, and using the Born rule at time $t = 3$ to compare the probabilities of the particle arriving at $3a$ or $3b$ in cases in which its path is detected by the $\hat{a}$ or by the $\hat{b}$ or by neither detector. That is, you should calculate conditional probabilities for $3a$ and $3b$, conditioned upon different detector outcomes. Then briefly discuss how this relates to Feynman's claim.

3. For each of the matrices given below, decide whether if suitable (possibly complex) numbers are inserted in the blank spaces marked $*$, the result is a density matrix (matrix of a density operator in an orthonormal basis). Explain what you are doing; give reasons for your answers.

$$\text{(i): } \begin{pmatrix} * & i/4 \\ -i/4 & 2/3 \end{pmatrix}, \quad \text{(ii): } \begin{pmatrix} (1+i)/4 & i/5 \\ 1/5 & * \end{pmatrix}, \quad \text{(iii): } \begin{pmatrix} 1/4 & 1/2 \\ * & 3/4 \end{pmatrix},$$

$$\text{(iv): } \begin{pmatrix} 1/5 & 0 & 0 \\ 0 & * & (1+i)/4 \\ * & (1-i)/4 & 1/3 \end{pmatrix}, \quad \text{(v): } \begin{pmatrix} * & 0 & 0 \\ 0 & 1/3 & i/4 \\ 0 & * & 1/6 \end{pmatrix}$$

4. a) Suppose that that all you know about a spin-half particle is that it was prepared in the state $|z^+\rangle$ with probability 1/3, in the state $|z^-\rangle$ with probability 1/3, and in the state $|y^+\rangle$ with probability 1/3. What density operator should you assign to it? Express your answer as a matrix in the standard $\{|z^+\rangle, |z^-\rangle\}$ basis, and also find the corresponding vector $\mathbf{r} = (x, y, z)$ in the Bloch sphere representation

$$\rho = \tfrac{1}{2}(I + x\sigma_x + y\sigma_y + z\sigma_z) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

b) Suppose that a measurement of σ_y is carried out on this spin-half particle. Obtain the probabilities for $\sigma_y = \pm 1$ by writing the density matrix as a matrix in the $|y^+\rangle$, $|y^-\rangle$ basis. Then calculate the average $\langle\sigma_y\rangle$ by evaluating $\mathrm{Tr}(\rho\sigma_y)$ in the standard basis. Are these results consistent? (If not, you have made a mistake.)

c) [Optional] Assuming a general spin-half density matrix ρ, find an expression for $\langle\sigma_y\rangle$ in terms of the Bloch sphere vector $\mathbf{r}$ associated with ρ.

5. a) Let $|\psi\rangle = \sum_{j,p} c_{jp}|a^j\rangle \otimes |b^p\rangle$ be an entangled ket on $\mathcal{H}_a \otimes \mathcal{H}_b$, expanded in a product of two orthonormal bases. Find an expression for $\rho_b = \mathrm{Tr}_a(|\psi\rangle\langle\psi|)$ as a sum of dyads with suitable coefficients.

b) Let Q be an operator,

$$Q = \sum_{jk}\sum_{pq}\langle a^j b^p|Q|a^k b^q\rangle \cdot |a^j b^p\rangle\langle a^k b^q|$$

on $\mathcal{H}_a \otimes \mathcal{H}_b$. Find the matrix elements $\langle a^j|Q_a|a^k\rangle$ and $\langle b^p|Q_b|b^q\rangle$ for the partial traces $Q_a = \mathrm{Tr}_b(Q)$ and $Q_b = \mathrm{Tr}_a(Q)$, and show that $\mathrm{Tr}_a(Q_a) = \mathrm{Tr}_b(Q_b) = \mathrm{Tr}(Q)$.

c) Let Q be any operator on $\mathcal{H}_a \otimes \mathcal{H}_b$, and let $A = A \otimes I$ and $B = I \otimes B$ be operators on $\mathcal{H}_a$ and $\mathcal{H}_b$, respectively. Show that

$$\mathrm{Tr}_b(AQ) = A\mathrm{Tr}_b(Q), \quad \mathrm{Tr}_b(BQ) = \mathrm{Tr}_b(QB).$$

Notice how the second of these generalizes the rule that products of operators can be cycled inside a trace.

d) [Optional] Is $\mathrm{Tr}_b(AQ) = \mathrm{Tr}_b(QA)$? Either prove it or find a counterexample.