33-445 Advanced Quantum Physics Fall Semester, 2012
Assignment No. 6. Due Friday, October 5

COURSE WEB PAGE:
http://quantum.phys.cmu.edu/quad/

READING: Sources

Townsend = *A Modern Approach to Quantum Mechanics*, 2d ed
CQT = Consistent Quantum Theory. Individual chapters at: http://quantum.phys.cmu.edu/CQT/
ROTAM = "Rotations and Angular Momentum" on course web page
UTDBR = "Unitary Time Development and Born Rule" on course web page

READING: Topics

Unitary time development: Townsend Ch. 4; CQT Ch. 7; UTDBR

Born rule: Although Townsend uses this, he does not give a clear statement of it; CQT Ch. 9 uses the language histories, but if you skip that part the rest should be accessible; UTDBR

Toy models: CQT Secs. 2.5, 6.3, 7.4; UTDBR

READING AHEAD:

Stochastic histories: CQT Ch. 8

Consistency conditions and probabilities of histories: CQT Ch. 11 and Sec. 11.6. We will not be considering the most general case in Ch.11, and will instead focus on chain kets, Sec. 11.6.

EXERCISES:

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course.

2. Consider the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ for the spin states of two (different) spin-half particles. Use the standard basis of eigenstates of S_{1z} and S_{2z}, of the form $|z^+z^-\rangle = |z^+\rangle \otimes |z^-\rangle$, etc., in the order $|z^+z^+\rangle, |z^+z^-\rangle, |z^-z^+\rangle, |z^-z^-\rangle$. Set $\hbar = 1$, so the eigenvalues of S_{1z} are $\pm 1/2$.

a) Work out what S_{1z} and S_{1x} do to each of the states in the standard basis, assuming that $S_x = \frac{1}{2}(|x^+\rangle\langle x^+| - |x^-\rangle\langle x^-|)$ and $|x^+\rangle = (|z^+\rangle + |z^-\rangle)/\sqrt{2}$, $|x^-\rangle = (|z^+\rangle - |z^-\rangle)/\sqrt{2}$.
For example, $S_{1z}|z^-z^+\rangle = -\frac{1}{2}|z^-z^+\rangle$.

b) Calculate the matrix elements of the operator $S_{1x} \otimes S_{2z}$ in the standard basis.

c) Construct a product state on $\mathcal{H}_1 \otimes \mathcal{H}_2$ which is both an eigenstate of S_{1x} with eigenvalue $+1/2$, and also an eigenstate of S_{2x} with eigenvalue $-1/2$.

d) Is $(|z^+z^-\rangle - |z^-z^+\rangle)$ a product state or an entangled state? Give reasons for your answer.

e) Write the projector for $S_{1x} = +1/2$ as a sum of dyads on $\mathcal{H}_1 \otimes \mathcal{H}_2$ constructed from the standard basis, i.e., things of the type

$$|z^+z^-\rangle\langle z^+z^+| = |z^+\rangle\langle z^+| \otimes |z^-\rangle\langle z^+|.$$

f) [Optional. Do not turn in, but think about it.] Find a projector on $\mathcal{H}_1 \otimes \mathcal{H}_2$ which is neither a product projector, of the form $P_1 \otimes P_2$, nor the sum $P_1 \otimes P_2 + P_1' \otimes P_2' + \cdots$ of product projectors, and indicate why you know that it is neither of these possibilities.

3. NOTE: Operators and eigenkets for a spin-1 particle in the standard basis can be found, among other places, in the most recent version of ROTAM, *Rotations and Angular Momentum*, on the course web page.

A spin-1 particle is placed in a magnetic field in the z direction leading to a Hamiltonian

$$H = \hbar\omega_0 S_z,$$

where S_z is dimensionless.

a) Find the unitary time development operator $U(t)$. Write it as a 3×3 matrix in the basis of eigenstates of S_z, in the order $|1\rangle$, $|0\rangle$, $|-1\rangle$. Use this matrix to find the column vector corresponding to $|\psi(t)\rangle = U(t)|\psi_0\rangle$, where

$$|\psi_0\rangle = \tfrac{1}{2}\bigl(|1\rangle + i\sqrt{2}|0\rangle - |-1\rangle\bigr).$$

b) Use $|\psi(t)\rangle$ as a pre-probability to find the probability distributions for S_x, S_y, and S_z at time t. That is, find the probabilities that $S_x = +1$, $S_x = 0$, and $S_x = -1$, and the same for S_y and S_z. Check that in each case the probabilities sum to 1.

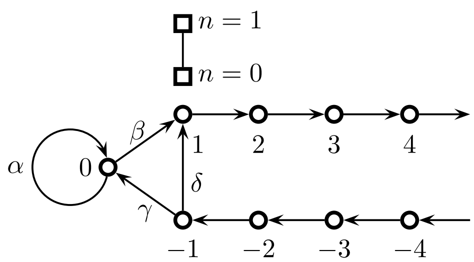
c) Use the probabilities calculated in (b) to construct the averages $\langle S_x\rangle_t$, $\langle S_y\rangle_t$, $\langle S_z\rangle_t$ as functions of time. You may want to check your work by seeing if $\langle S_x\rangle_t$ obtained using the probabilities in (b) is the same as $\langle\psi(t)|S_x|\psi(t)\rangle$ obtained using $|\psi(t)\rangle$ and the matrix for S_x.

d) Use the formula relating $d\langle A\rangle_t/dt$ to $[H, A]$ in order to relate $d\langle S_x\rangle_t/dt$ and $d\langle S_y\rangle_t/dt$ to $\langle S_x\rangle_t$ and $\langle S_y\rangle_t$. Check that these are consistent with your answer to (c).

4. A toy model of alpha decay is shown in the figure, along with a two-state detector. The Hilbert space of the particle plus detector is $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_d$. The unitary time development operator is

$$T = RS,$$

where S acting on $\mathcal{H}_p$ is the unitary shift operator, $S|m\rangle = |m+1\rangle$, except for the periodic boundary condition $S|M_b\rangle = |-M_a\rangle$ — assume M_a, M_b, and $M = 1 + M_a + M_b$ are all large numbers — and also



$$S|0\rangle = \alpha|0\rangle + \beta|1\rangle, \quad S|-1\rangle = \gamma|0\rangle + \delta|1\rangle,$$

with coefficients chosen so that $U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ a unitary matrix. In addition, $R|m, n\rangle = |m, n\rangle$ with the exception that when $m = 1$,

$$R|1, n\rangle = |1, 1-n\rangle.$$

Assume an initial state at time $t = 0$ with $m = 0$, the alpha particle inside the nucleus, and $n = 0$, the detector in its ready state:

$$|\psi_0\rangle = |0, 0\rangle = |0\rangle \otimes |0\rangle.$$

a) Write down a projector P whose physical interpretation is that the particle is at $m = 0$ (the nucleus has not yet decayed), and another projector Q whose interpretation is that the detector is in the triggered state $n = 1$. Both operators act on the tensor product space $\mathcal{H}_p \otimes \mathcal{H}_d$ and should be written using $\otimes$ symbols to make it perfectly clear what they mean.

b) Work out

$$|\psi_t\rangle = T^t|\psi_0\rangle$$

for $t = 1, 2, \ldots$; enough terms so that you can see the pattern that emerges. Then use the Born rule and the projectors you found in (a) to write down expressions for the following probabilties, at any (integer) time $t > 1$ but not too large: (i) the probability that the nucleus has not yet decayed, (ii) the probability that the detector has triggered, and (iii) the probability that the nucleus has not decayed AND that the detector has triggered. (Note that the unitarity of U implies that $|\alpha|^2 + |\beta|^2 = 1$.) Do your results make physical sense?