33-445 Advanced Quantum Physics Fall Semester, 2012
Assignment No. 5. Not to be turned in

ANNOUNCEMENT: There will be an hour exam during class the morning of Friday, Sept. 28.

The instructor will be present and you can begin the examination at 9:20. All exam booklets must be turned in by 10:30.

The examination will be closed book, closed notes, no pocket calculators. Bring a sharp pencil. The examination will be on the material covered in class through Wednesday, Sept. 19, and found in assignments 1 to 4. This includes the material in Chs. 1 to 3 of Townsend, and in CQT Chs. 3 (except Sec. 3.9), Ch. 4, and Ch. 5, Secs. 5.1, 5.2, 5.5, and 5.6. Also the notes "Hilbert Space Quantum Mechanics," "Probabilities," "Rotations and Angular Momentum". Material in the lectures devoted to composite systems, tensor products, toy models, unitary time development is *not* included in this examination.

COURSE WEB PAGE:
http://quantum.phys.cmu.edu/quad/

READING: Sources
Townsend = *A Modern Approach to Quantum Mechanics*, 2d ed
CQT = Consistent Quantum Theory. Individual chapters at: http://quantum.phys.cmu.edu/CQT/

READING: Topics
Composite systems and tensor products: Townsend Secs 5.1 and 5.2; CQT Ch. 6
Toy models: CQT Secs. 2.5, 6.3, 7.4
Unitary time development: Townsend Ch. 4; CQT Ch. 7

EXERCISES:

The following are intended to help you review for the hour exam. They are *not to be turned in.* Solutions will be provided on Wednesday, Sept. 26.

1. A certain quantum property is represented by a projector

$$Q = \begin{pmatrix} 1/6 & i/3 & -i/6 \\ -i/3 & 2/3 & -1/3 \\ i/6 & -1/3 & 1/6 \end{pmatrix}.$$

in an orthonormal basis.

a) What properties must Q possess to be a projector? Indicate how they are satisfied by this matrix.

b) Consider a decomposition of the identity $\{P_1, P_2\}$ with $P_1 = Q$. What is P_2? What are the dimensions of the subspaces onto which P_1 and P_2 project?

c) Given the pre-probability

$$|\psi\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

what is the probability of P_1? of P_2?

d) Same question in the case where (note that this ket is not normalized):

$$|\psi\rangle = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix},$$

2. Let $\{|b_j\rangle\}$ and $\{|c_l\rangle\}$ be two orthonormal bases of the Hilbert space, and define the two matrices

$$A_{jk} := \langle b_j|A|b_k\rangle, \quad \hat{A}_{lm} = \langle c_l|A|c_m\rangle$$

corresponding to the same operator A. The matrices are related by

$$\hat{A} = VAW, \text{ or } \hat{A}_{lm} = \sum_{j,k} V_{lj} A_{jk} W_{km}.$$

with a suitable choice of matrices V and W.

a) What kind of matrices (operators) are V and W, and how are they related to each other?

b) Find explicit expressions for the matrix elements V_{lj} and W_{km} in terms of the two bases, and use these expressions to confirm your answer to (a). [Hint. Would completeness help?]

3. Let $|\frac{3}{2}, m\rangle$ be a ket corresponding to spin (or total angular momentum) $j = 3/2$, with $J_z|\frac{3}{2}, m\rangle = m|\frac{3}{2}, m\rangle$.

a) What are the possible value of m? (Just state the answer; you do not have to justify it.)

b) Let $J_+ = J_x + iJ_y$, $J_- = J_x - iJ_y$. There is a state $|\frac{3}{2}, m\rangle$ such that J_+ applied to this state gives 0, and a different state $|\frac{3}{2}, m'\rangle$ such that J_- applied to it gives 0. What are m and m'? You need not give an explanation provided you confirm the correctness of your answer in (e) below.

c) Express J_-J_+ as a sum with suitable coefficients of the operators J_z, J_z^2, and $J^2(= J_x^2 + J_y^2 + J_z^2)$, J_z.

d) Show how your expression for J_-J_+ in (c) can be used to compute $|c_m|^2$, where c_m occurs in the relationship

$$J_+|\tfrac{3}{2}, m\rangle = c_m|\tfrac{3}{2}, m+1\rangle,$$

and $|\frac{3}{2}, m\rangle$ and $|\frac{3}{2}, m+1\rangle$ are assumed to be normalized.

e) Use the result in (d) to confirm your answer to (b).

4. Consider a particle of spin $j = 1$. The rotation operator for an angle ω about the y axis may be written as:

$$R_y(\omega) = \exp[-i\omega J_y],$$

where J_y is the y component of the (dimensionless) angular momentum operator $\vec{J}$.

a) Let $|\psi_1\rangle$ be a normalized eigenket of J_x with eigenvalue $+1$, and $|\psi_2\rangle$ a normalized eigenket of J_z with eigenvalue -1. Argue that $|\psi_2\rangle$ can be obtained from $|\psi_1\rangle$ by suitable use of $R_y(\omega)$ with an appropriate choice of ω, and similarly $|\psi_1\rangle$ can be obtained from $|\psi_2\rangle$, perhaps using a different value of ω. Might there be a problem with an overall phase?

b) Let P_1 and P_2 be projectors onto the rays (one-dimensional subspaces) corresponding to $|\psi_1\rangle$ and $|\psi_2\rangle$. How can you use a $R_y(\omega)$ rotation operator to obtain P_2 from P_1, and vice versa? Might there be a problem with an overall phase?

c) Willy Smart wants to construct a decomposition of the identity $\{P_1, P_2, P_3\}$ with the P_1 and P_2 the projectors in (b), along with a suitable P_3. Discuss.