

COURSE WEB PAGE:

<http://quantum.phys.cmu.edu/quad/>

READING: Sources

Townsend = *A Modern Approach to Quantum Mechanics*, 2d ed

CQT = Consistent Quantum Theory. Individual chapters at: <http://quantum.phys.cmu.edu/CQT/>

HSQM = “Hilbert Space Quantum Mechanics” on course web page

PROBS = “Probabilities” on course web page

READING: Topics

Indicator functions: CQT Sec. 4.1

Decomposition of the identity: HSQM Sec. 2.9, 2.10; PROBS Sec. 2, 3; CQT Sec. 3.5

Physical properties: CQT Ch. 4

Physical variables: CQT Sec. 5.5

Born rule for probabilities: PROBS Sec. 5

Unitary operators: Townsend Ch. 2; HSQM Sec. 2.8; CQT Sec. 7.2

Rotations in three dimensions: Townsend Ch. 3

Spin one: Townsend Sec. 3.7

Angular momentum (mostly next week): Townsend Ch. 3.

READING AHEAD:

Angular momentum: Townsend Ch. 3

Unitary time development: Townsend Ch. 4; CQT Ch. 7

EXERCISES:

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course.

2. The matrices

$$Q = \begin{pmatrix} 1/6 & i/3 & -i/6 \\ -i/3 & 2/3 & -1/3 \\ i/6 & -1/3 & 1/6 \end{pmatrix} \quad R = \begin{pmatrix} 1/2 & 0 & -i/2 \\ 0 & 1 & 0 \\ i/2 & 0 & 1/2 \end{pmatrix}$$

represent two commuting projectors on a three-dimensional Hilbert space \mathcal{H} . Find the (unique) decomposition $\{P_j\}$ of the identity I such that Q and R belong to the corresponding Boolean event algebra, which is to say they can be written as sums of one or more of the P_j . Express your answer in terms of 3×3 matrices using the same basis as for Q and R above. (Hints: If B and C are members of the Boolean event algebra of projectors, then so are BC , $I - B$, and $I - C$. How is the trace of a projector related to the dimension of the subspace on which it projects?)

3. a) Let S and T be two indicator functions on the classical phase space, and suppose that $ST = T$. One of the following assertions can be proved.

(i) If the system possesses property S it necessarily also possesses property T .

(ii) If the system possesses property T it necessarily also possesses property S .

Decide which is correct and explain why (a formal proof is not needed).

b) For a 3-dimensional Hilbert space construct two projectors P and Q that do not commute, $PQ \neq QP$, with the property that the intersection of $\mathcal{R} = \mathcal{P} \cap \mathcal{Q}$ of the subspaces \mathcal{P} and \mathcal{Q} onto which they project is of dimension 1. Try and make it a very simple example. Let R be the projector onto \mathcal{R} . What are PR and QR ? [Hint 1: Both are projectors. Hint 2: You may prefer to work backwards, starting from R or \mathcal{R} .]

c) Assuming that classical mechanics, i.e. (a), is a suitable guide, what might you conclude about relationships of the properties of a quantum system on the basis of what you found in (b)? Why might this conclusion be somewhat troubling in light of the fact that $PQ \neq QP$?

[Hint for the following problem: Sec. 3.7 in Townsend discusses spin 1.]

4. For a spin-one particle the matrix of S_y (setting $\hbar = 1$) in the standard of S_z basis is

$$S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

a) Find the projectors in the decomposition $\{P_j\}$ corresponding to S_y (i.e., $S_y = \sum_j s_j P_j$, writing each projector as a matrix in the standard basis. (Check that the matrices are Hermitian and sum to the identity.) [Hint: Can you find the eigenvectors of S_y ?]

b) Calculate the Born rule probabilities $\langle \psi | P_j | \psi \rangle$ for the different eigenvalues of S_y assuming a pre-probability

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}.$$

That is, first compute the column vector $P_j |\psi\rangle$ and then find its inner product with the row vector $\langle \psi |$. Check that the probabilities add up to 1.

c) Calculate $\langle S_y \rangle$ using the eigenvalues of S_y and the probabilities you computed in (b). Then calculate $\langle \psi | S_y | \psi \rangle$ as a row vector times a matrix times a column vector. The two results should be the same.

5. a) Let A and B be two matrices corresponding to commuting Hermitian operators, $[A, B] = 0$. Assume that the matrix of A is of the form

$$A = \begin{pmatrix} a & b & c \\ b^* & d & e \\ c^* & e^* & f \end{pmatrix}.$$

Of course a , d , and f are real numbers. What else can you say in each of the following cases, based upon $[A, B] = 0$?

$$(i): B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad (ii): B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

b) There is a theorem that says that if A and B are two Hermitian (more generally, normal) operators that commute with each other it is always possible to find an orthonormal basis in which both are diagonal. Discuss how what you found in (a) is consistent with this theorem.

6. Let the matrix elements of an operator A be $A_{jk} = \langle b_j | A | b_k \rangle$ in the orthonormal basis $\{|b_j\rangle\}$ and $\bar{A}_{lm} = \langle c_l | A | c_m \rangle$ in the orthonormal basis $\{|c_l\rangle\}$.

a) Show that the A and \bar{A} matrices can be related by a unitary matrix U , $\bar{A} = UAU^\dagger$, where the matrix U can be expressed in terms of elements of the two orthonormal bases.

b) Suppose that in the $\{|z^+\rangle, |z^-\rangle\}$ basis for a spin half particle the matrix of A is

$$A = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}.$$

Use the construction in (a) to find the matrix \bar{A}_{lm} in the

$$|y^+\rangle = (|z^+\rangle + i|z^-\rangle)/\sqrt{2}, \quad |y^-\rangle = (|z^+\rangle - i|z^-\rangle)/\sqrt{2}$$

basis—i.e., construct the matrix U and then calculate the matrix product UAU^\dagger .

c) Would it make a difference in the \bar{A} matrix if we used a different phase for $|y^-\rangle$ than that employed in (b)? E.g., suppose that

$$|y^-\rangle = e^{i\phi}(|z^+\rangle - i|z^-\rangle)/\sqrt{2}.$$

How does this change things?