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Scattering Amplitudes for (Almost) Identical Particles

The textbook treatment of the scattering of identical particles, as in Taylor Ch. 22, leads to expressions in which the scattering amplitude for, say, identical bosons is the sum of two terms

$$f(\Omega) + f(\bar{\Omega}) \qquad (1a)$$

$$\Omega = (\theta, \phi) \qquad \bar{\Omega} = (\pi - \theta, \phi + \pi) \qquad (1b)$$

and in the ~~oth~~ antisymmetric case, e.g., two spin 1/2 fermions in a singlet state, to the difference

$$f(\Omega) - f(\bar{\Omega})$$

It is then natural to ask for the meaning of $f(\Omega)$ by itself. What it seems to represent is the result of an idealized calculation in which two <u>nonidentical</u> particles which have, nonetheless, identical physical properties (same mass, same charge, same interactions) interact with and scatter from each other. I call these "almost identical" or AI particles

Let the AI particles be 1 and 2, with positions $\vec{r}_1$, $\vec{r}_2$, and define the usual center of mass

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \quad = (x, y, z) \qquad (2a)$$

$$\vec{R} = \tfrac{1}{2}(\vec{r}_1 + \vec{r}_2) \qquad (2b)$$

We hereafter ignore $\vec{R}$, which is conveniently symmetrical under interchange of particle labels. The coordinate $\vec{r}$ represents a single "pseudoparticle". Let us imagine that we are dealing with simple potential scattering. Then it is plausible that there is an appropriate solution to the (time-independent) Schr. equation

$$\psi(\vec{r}) \qquad (2c)$$

with asymptotic behavior

$$\psi(\vec{r}) \simeq e^{ikz} + \frac{f(\Omega)}{r} e^{ikr}, \text{ large } r \qquad (2d)$$

with

$$z = z_1 - z_2 \qquad r = |\vec{r}|$$

and scattering amplitude $f(\;)$ described in terms of this asymptotic behavior

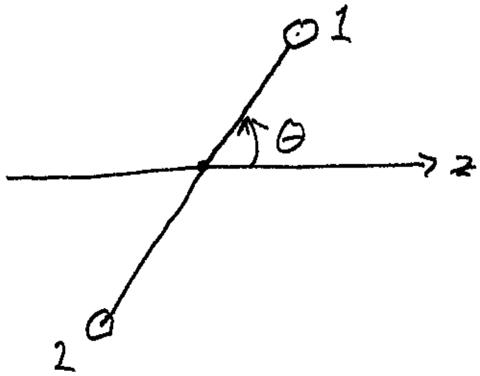
Let us shift notation slightly and define

$$\Psi(\vec{r_1}, \vec{r_2}) = \psi(\vec{r_1} - \vec{r_2}) \tag{3a}$$

to emphasize that we are dealing with two particles, in which case (2d) becomes an asymptotic form

$$\Psi(\vec{r_1}, \vec{r_2}) \simeq e^{ik(z_1 - z_2)} + \frac{f(\Omega_{12})}{|\vec{r_1} - \vec{r_2}|} e^{ik|\vec{r_1} - \vec{r_2}|} \tag{3b}$$

and Ω_{12} is the pair of polar angles (θ, ϕ) with describes particle 1 as seen from particle 2



Note that $\Psi(\vec{r_1}, \vec{r_2}) \neq \Psi(\vec{r_2}, \vec{r_1})$ and for the latter

$$\Psi(\vec{r_2}, \vec{r_1}) \simeq e^{ik(z_2 - z_1)} + \frac{f(\bar{\Omega}_{12}) e^{ik|\vec{r_1} - \vec{r_2}|}}{|\vec{r_1} - \vec{r_2}|}$$

and

$$\Omega_{21} = \bar{\Omega}_{12}$$

is obtained (1b), by $\theta \to \pi - \theta$, $\phi \to \pi + \phi$

We may now form symmetrical and antisymmetrical combinations

$$\Psi_S(\vec{r}_1, \vec{r}_2) = \Psi(\vec{r}_1, \vec{r}_2) + \Psi(\vec{r}_2, \vec{r}_1) \qquad (4a)$$

$$\Psi_A(\vec{r}_1, \vec{r}_2) = \Psi(\vec{r}_1, \vec{r}_2) - \Psi(\vec{r}_2, \vec{r}_1) \qquad (4b)$$

The asymptotic behavior is

$$\Psi_S(\vec{r}_1, \vec{r}_2) = \left[e^{ik(z_1 - z_2)} + e^{ik(z_2 - z_1)} \right] + \frac{\left[f(\Omega_{12}) + f(\bar{\Omega}_{12}) \right]}{|\vec{r}_1 - \vec{r}_2|} e^{ik|\vec{r}_1 - \vec{r}_2|} \qquad (4c)$$

$$\Psi_A(\vec{r}_1, \vec{r}_2) = \left[e^{ik(z_1 - z_2)} - e^{ik(z_2 - z_1)} \right] + \frac{\left[f(\Omega_{12}) - f(\bar{\Omega}_{12}) \right]}{|\vec{r}_1 - \vec{r}_2|} e^{ik|\vec{r}_1 - \vec{r}_2|} \qquad (4d)$$

Let us <u>define</u>

$$f_S(\Omega) = f(\Omega) + f(\bar{\Omega}) \qquad (4e)$$

$$f_A(\Omega) = f(\Omega) - f(\bar{\Omega}) \qquad (4f)$$

Then the usual arguments relating amplitudes to cross section give us, but see later comments,

$$\sigma_S(\Omega) = |f_S(\Omega)|^2 = |f(\Omega) + f(\bar{\Omega})|^2 \qquad (4g)$$

$$\sigma_A(\Omega) = |f_A(\Omega)|^2 = |f(\Omega) - f(\bar{\Omega})|^2 \qquad (4h)$$

The following is a confusing point that needs to be addressed. We have computed $\sigma_s(\Omega)$ by following the usual prescription of taking an amplitude and then its absolute square. But would it not have been more reasonable to, say, define

$$\Psi_s(\vec{r}_1,\vec{r}_2) = [\Psi(\vec{r}_1,\vec{r}_2) + \Psi(\vec{r}_2,\vec{r}_1)]/\sqrt{2} \quad (5a)$$

in which case we would have arrived at

$$\sigma_s(\Omega) = \frac{1}{2}\,|f(\Omega)+f(\bar{\Omega})|^2 \ ? \quad (5b)$$

in place of (4g).

The issue here is a matter of <u>convention</u>, and can be cleared up by means of an example. Suppose that in the laboratory a beam of α particles is being scattered off a ^4He gas target at energies high enough so we can ignore the electrons attached to the He atoms. Then the α particles that emerge and are detected come both from the incoming beam and from the target, so that, assuming elastic collisions, <u>twice as many</u> alpha particles scatter outwards (integrating over all angles) as arrive in the incoming beam. However, the experimentalists may still (plausibly) divide the outgoing flux by their incoming beam current in <u>defining</u> the differential cross section, which will then be given correctly by (4g), and NOT by (5b).

This "overcounting" effect would also arise in the case of purely classical scattering of hard spheres by hard spheres of identical size, and imagining the classical situation may help separate out this "factor of 2" effect from the more interesting genuinely quantum mechanical mysteries.

<u>Effects of spin.</u>

When scattering of identical particles with internal degrees of freedom, say electrons with spin, is considered, these additional degrees of freedom may make the whole problem a lot more complicated (e.g., scattering leaves the beam or target nucleus in an excited state) ~~or it can be~~ or the effect can be rather trivial, related to statistics. We consider the simple case here.

Consider a case in which the particles have spin, but the spin is not altered during the scattering process and has no effect upon the scattering potential. In this case we work out what happens to almost identical (AI) spinless particles and find $f(\Omega)$. Then we address the spin problem in the manner discussed below.

To be specific, consider two spin-half particles, and let $\chi(\sigma_1, \sigma_2)$ denote the joint spin state: $\sigma_1 = \pm 1$, $\sigma_2 = \pm 1$ (or $\pm 1/2$ if preferred). Then there are two types of state with well-defined symmetry under the exchange of particle labels, the singlet and triplet states which we here denote by

$$\chi_{0,0} \quad , \quad \chi_{1m} \quad m = -1, 0, 1 \qquad (12a)$$

or by

$$|00\rangle \qquad |1m\rangle, \; m = -1, 0, 1 \qquad (12b)$$

The overall state of the two electrons has to be antisymmetric, which means we can form states of the form, see (4a, b)

$$|\Psi_S\rangle |00\rangle = \Psi_S \, \chi_{00} \qquad (12c)$$

or

$$|\Psi_A\rangle |1m\rangle = \Psi_A \, \chi_{1m} \qquad (12d)$$

where one can include the arguments

$$\vec{r}_1, \sigma_1 = q_1 \qquad \vec{r}_2, \sigma_2 = q_2$$

if desired, but this is more messy.

Note that the assumption that the spins play no role in the scattering is needed to justify the product forms in (12c, d) where the spin state is the same in incoming & outgoing waves.

If, then, the ~~initial~~ initial spin state of the two electrons is a singlet ~~state~~ χ_{00}, then the outgoing spin state will be the same and the scattering amplitude ~~so far~~ will be $f_S(\Omega)$, symmetric case, (4e). If the initial spin state is one of the triplet states χ_{1m}, then the scattering amplitude will be given by the asymmetrical $f_A(\Omega)$.

But it is very difficult to arrange in an experiment to have the beam and target spins in a singlet state, and often one is interested in cases in which both spins are "random". Knowing the symmetric and anti-symmetric scattering amplitudes $f_S(\Omega)$, $f_A(\Omega)$, one can work out the other cases in terms of them. It helps to look at examples

Conceptually the simplest situation is probably that in which both <u>target and beam particles</u> (electrons) <u>are polarized in the same direction</u>, say $\sigma_1 = \sigma_2 = +1$. Then since the scattering does not change the spin, the outgoing particles have the same spin orientation. We then have the case of an antisymmetric spatial function, so the scattering amplitude will be $f_A(\Omega)$, (4f), and the differential cross section will be $|f_A(\Omega)|^2$.

$$\sigma(\Omega) = |f_A(\Omega)|^2 = |f(\Omega) - f(\bar{\Omega})|^2, \quad (14a)$$

Another (fairly) simple case is that in which the <u>beam and target</u> particles are <u>polarized</u> in the <u>opposite</u> direction. We can think about this in two different ways.

Naive approach. We are dealing with a situation of <u>distinguishable</u> particles, so that the "experimental" cross section if the spin states are not distinguished by the detection apparatus will be

$$\sigma(\Omega) = |f(\Omega)|^2 + |f(\bar{\Omega})|^2, \quad (14b)$$

the first due to outgoing particles with same spin as incoming particles; the second due to target particles scattered into the detector

An alternative attack on the problem is to note that the initial spin state is

$$|+,-\rangle = \frac{1}{2}\left(|+,-\rangle - |-,+\rangle\right) + \frac{1}{2}\left(|+,-\rangle + |-,+\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\chi_{00} + \frac{1}{\sqrt{2}}\chi_{10} \tag{15a}$$

Thus what we need is an overall state of the form

$$\frac{1}{\sqrt{2}}\chi_{00}\Psi_S + \frac{1}{\sqrt{2}}\chi_{10}\Psi_A \tag{15b}$$

with asymptotic outgoing part

$$\left[\frac{1}{\sqrt{2}}\chi_{00} f_A(\Omega) + \frac{1}{\sqrt{2}}\chi_{10} f_S(\Omega)\right]\frac{1}{r}e^{ikr} \tag{15c}$$

Therefore the cross section will be given by

$$\cancel{\sigma(\Omega) = \frac{1}{2}\sigma_A(\Omega) + \frac{1}{2}\sigma_S(\Omega)} \quad \sigma(\Omega) = \frac{1}{2}\left[\sigma_A(\Omega) + \sigma_S(\Omega)\right]$$

$$= \frac{1}{2}\left[|f(\Omega) - f(\bar{\Omega})|^2 + |f(\Omega) + f(\bar{\Omega})|^2\right] \tag{15d}$$

$$= |f(\Omega)|^2 + |f(\bar{\Omega})|^2$$

since the cross terms cancel. The end result is the same as for the naive approach: (15d) the same as (14b)

We may also ask for what happens if both the incoming beam and the target are <u>unpolarized</u>.

The naive approach is to say: suppose the target is polarized but the incoming beam is not polarized. Then we should average the results obtained in (14a) and (14b), which is to say

$$\sigma(\Omega) = \frac{1}{2}\left|f(\Omega)-f(\bar\Omega)\right|^2 + \frac{1}{2}\left|f(\Omega)\right|^2 + \frac{1}{2}\left|f(\bar\Omega)\right|^2$$

$$= \left|f(\Omega)\right|^2 + \left|f(\bar\Omega)\right|^2 - \mathrm{Re}\left[f^*(\Omega)\, f(\bar\Omega)\right]$$

Whereas we assumed the target was polarized, the polarization direction does not enter into this cross section, so it is also correct if the target is unpolarized.

An alternative approach is to say: unpolarized beam and target means that 1/4 of the time the initial (and thus final) spin state is a singlet, 3/4 of the time it is a triplet, and therefore

$$\sigma(\Omega) = \frac{1}{4}\left|f(\Omega)+f(\bar\Omega)\right|^2 + \frac{3}{4}\left|f(\Omega)-f(\bar\Omega)\right|^2$$

$$= \left|f(\Omega)\right|^2 + \left|f(\bar\Omega)\right|^2 - \mathrm{Re}\left[f^*(\Omega)\, f(\bar\Omega)\right]$$