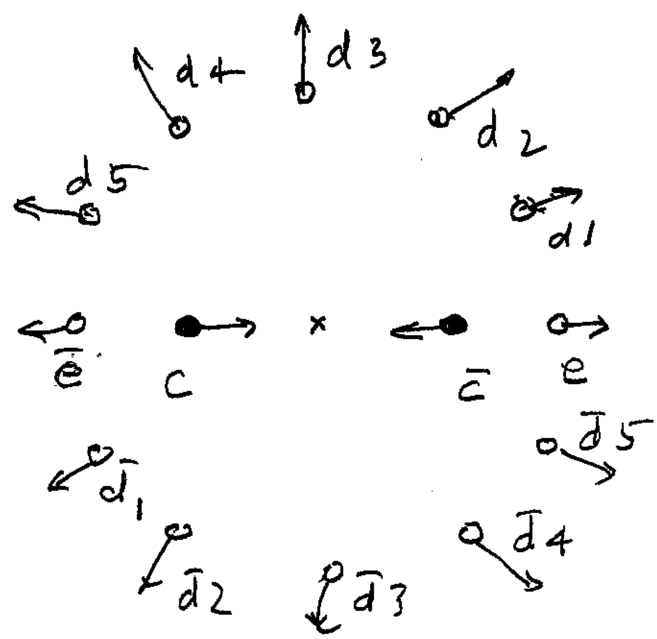
Toy model for scattering of identical particles

I wish to preserve some of the symmetry that is involved in the usual picture of two-particle scattering: zero momentum in the center of mass, particles coming in from both sides and allowed to scatter off each other; "direct" and "exchange" terms.

It would be nice to have a model in which one can employ the field-theoretic picture as well (creation and annihilation operators)

As per the following figure, imagine two input states for AI particles, labeled $c, \bar{c}$, wave packets traveling towards each other, and then pairs of output states d_j and $\bar{d}_j$, where these represent wavepackets symmetrically placed traveling outwards in opposite directions; $d0$ and $\bar{d}0$ are output states if no scattering occurs



(2a)

[Note: Read $d2$ as d_2, $\bar{d}2$ as $\bar{d}_2$, etc.]

Remark: Particle initially at c can scatter to <u>either d_j or $\bar{d}_j$</u> state for $j \geq 0$. However, if $c \to \bar{d}_2$, then necessarily $\bar{c} \to d_2$; if $c \to d2$ then necessarily $\bar{c} \to \bar{d}2$

Adopt notation appropriate to AI (almost identical) particles which are nonetheless regarded as **distinguishable**, and write

$$|c\bar{c}\rangle = |1{:}c, 2{:}\bar{c}\rangle = |1c2\bar{c}\rangle$$

where the : notation is Cohen-Tannoudji et al, and $1{:}c, 2{:}\bar{c}$ means particle 1 is in (orbital) c, and particle 2 in $\bar{c}$. The $|c\bar{c}\rangle$ notation requires that put the labels inside the ket ~~in the~~ same order as ~~the~~ ~~parcels~~ particles, so $|c\bar{c}\rangle$ is not the same thing as $|\bar{c}c\rangle$; indeed, they are orthogonal to each other assuming that the orbitals are orthogonal,

$$\langle c|\bar{c}\rangle = 0$$

Then one can define symmetric and antisymmetric states

$$|c\bar{c}, S\rangle = |c\bar{c}\rangle_S = \frac{1}{\sqrt{2}}\left(|c\bar{c}\rangle + |\bar{c}c\rangle\right)$$

$$|c\bar{c}, A\rangle = |c\bar{c}\rangle_A = \frac{1}{\sqrt{2}}\left(|c\bar{c}\rangle - |\bar{c}c\rangle\right)$$

Time development. In terms of (2a), and using d_2 rather than d2 (to avoid confusion with particle labels, we assume that

$$T|c\bar{c}\rangle = T|1c, 2\bar{c}\rangle = \beta_1|d_1\bar{d}_1\rangle + \bar{\beta}_1|\bar{d}_1 d_1\rangle \qquad (3a)$$
$$+\beta_2|d_2\bar{d}_2\rangle + \bar{\beta}_2|\bar{d}_2 d_2\rangle + \cdots + \gamma|e\bar{e}\rangle + \bar{\gamma}|\bar{e}e\rangle$$

where on the rhs the full symbols are, for example,

$$|\bar{d}_1 d_1\rangle = |1\bar{d}_1, 2d_1\rangle \qquad (3b)$$

i.e., particle 1 is in $\bar{d}_1$ and particle 2 is in d_1

Notice that there is no reason to expect that β_1 will have any simple relationship to $\bar{\beta}_1$. E.g., if scattering is mostly in the forwards direction one would expect, in terms of (2a), ~~that~~

$$|\beta_1| \gg |\bar{\beta}_1|$$

Naturally, given AI particles, we expect that if we start off with particle 2 at c and 1 at $\bar{c}$, the counterpart of (3a) is

$$T|\bar{c}c\rangle = T|1\bar{c}, 2c\rangle = \beta_1|\bar{d}_1 d_1\rangle + \bar{\beta}_1|d_1\bar{d}_1\rangle$$
$$+\beta_2|\bar{d}_2 d_2\rangle + \bar{\beta}_2|d_2\bar{d}_2\rangle + \cdots + \gamma|\bar{e}e\rangle + \bar{\gamma}|e\bar{e}\rangle$$

In order to see the essentials without too much distraction, let us assume that all the β_j and $\bar\beta_j$ are zero except for $j=1$, and write

$$\beta_1=\beta,\quad \bar\beta_1=\bar\beta,\qquad \beta_2=\beta_3=\cdots=0=\bar\beta_2=\bar\beta_3=\cdots \qquad (4a)$$

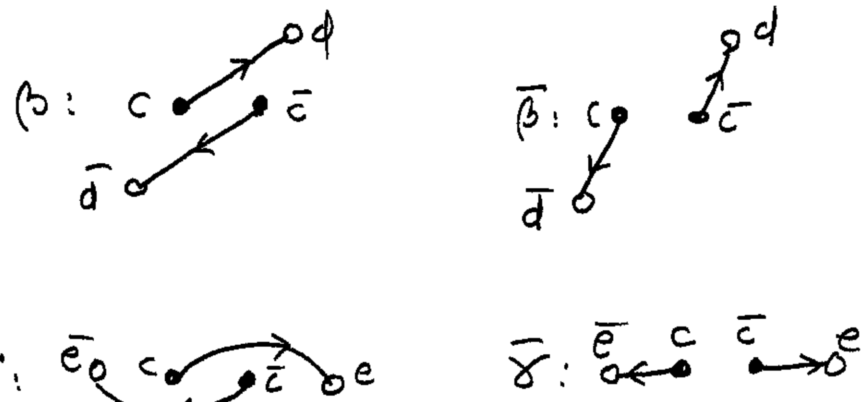
$$d_1=d\quad \bar d_1=\bar d \qquad (4b)$$

giving us then the simpler expressions

$$T|c\bar c\rangle=\beta|d\bar d\rangle+\bar\beta|\bar d d\rangle+\gamma|e\bar e\rangle+\bar\gamma|\bar e e\rangle \qquad (4c)$$

$$T|\bar c c\rangle=\beta|\bar d d\rangle+\bar\beta|d\bar d\rangle+\gamma|\bar e e\rangle+\bar\gamma|e\bar e\rangle \qquad (4d)$$

It may help to draw diagrams so as to "see" the meanings of these amplitudes



(4e)

$\boxed{\text{Unitarity.}}$ Since $|c\bar c\rangle$ and $|\bar c c\rangle$ are orthogonal and assumed normalized, (4c) and (4d) together imply

$$|\beta|^2+|\bar\beta|^2+|\gamma|^2+|\bar\gamma|^2=1 \qquad (4f)$$

$$\beta\bar\beta^*+\beta^*\bar\beta+\gamma\bar\gamma^*+\gamma^*\bar\gamma=0 \qquad (4g)$$

where the second can be written as

$$\mathrm{Re}(\beta\bar\beta^*)+\mathrm{Re}(\gamma\bar\gamma^*)=0 \qquad (4h)$$

Now let us ask what happens if we apply T to (anti) symmetrized states:

$$T|c\bar{c}\rangle_S = \frac{1}{\sqrt{2}}\left(T|c\bar{c}\rangle + T|\bar{c}c\rangle\right)$$

$$= \beta|d\bar{d}\rangle_S + \bar{\beta}|d\bar{d}\rangle_S + \gamma|e\bar{e}\rangle_S + \bar{\gamma}|e\bar{e}\rangle_S \qquad (5a)$$

$$= (\beta+\bar{\beta})|d\bar{d}\rangle_S + (\gamma+\bar{\gamma})|e\bar{e}\rangle_S$$

using the fact that

$$|d\bar{d}\rangle_S = |\bar{d}d\rangle_S = \frac{1}{\sqrt{2}}\left(|d\bar{d}\rangle + |\bar{d}d\rangle\right) \qquad (5b)$$

and similarly for $|e\bar{e}\rangle_S$. Summary:

$$T|c\bar{c}\rangle_S = (\beta+\bar{\beta})|d\bar{d}\rangle_S + (\gamma+\bar{\gamma})|e\bar{e}\rangle_S \qquad (5c)$$

In the same way,

$$T|c\bar{c}\rangle_A = (\beta-\bar{\beta})|d\bar{d}\rangle_A + (\gamma-\bar{\gamma})|e\bar{e}\rangle_A \qquad (5d)$$

where one has to pay attention to the sign convention, which tells us that

$$|d\bar{d}\rangle_A = -|\bar{d}d\rangle_A = \frac{1}{\sqrt{2}}\left(|d\bar{d}\rangle - |\bar{d}d\rangle\right) \qquad (5e)$$

Let us now calculate some probabilities, starting with the AI situation represented in, say, (4c). The Born rule gives, where subscript D = "distinguishable"

$$Pr_D(d\bar{d}\,|\,c\bar{c}) = |\beta|^2 \quad Pr_D(\bar{d}d\,|\,c\bar{c}) = |\bar{\beta}|^2$$
$$Pr_D(e\bar{e}\,|\,c\bar{c}) = |\gamma|^2 \quad Pr_D(\bar{e}e\,|\,c\bar{c}) = |\bar{\gamma}|^2 \qquad (6a)$$

and the sum of these is 1 by (4f). This makes sense: the particles are distinguishable through AI, and if particle 1 starts in c, particle 2 in $\bar{c}$, then the probability that after scattering particle 1 will be found at d, particle 2 at $\bar{d}$, is $|\beta|^2$, which is (in general) different from particle 1 at $\bar{d}$ and particle 2 at d.

If, on the other hand, we ignore (for purposes of computation) the difference, and ask "what is the probability that there will be a particle at d and a particle at $\bar{d}$", this will be

$$Pr_D(d\bar{d}\,|\,c\bar{c}) + Pr_D(\bar{d}d\,|\,c\bar{c}) = |\beta|^2 + |\bar{\beta}|^2 \qquad (6b)$$

and of course the same answer is obtained if we assume an initial state $\bar{c}c$: particle 1 in $\bar{c}$ and particle 2 in c.

In the case of identical and not just AI particles, it is sensible to ask for the probability of a particle at d and a particle at $\underline{d}$ and not whether particle 1 is at d and 2 at $\underline{d}$, as the particles cannot be distinguished. These probabilities can be computed from the formulas on. 5; in particular

$$\Pr_S(d\bar{d}\,|\,c\bar{c}) = |\beta+\bar{\beta}|^2 = |\beta|^2 + |\bar{\beta}|^2 + 2\,\mathrm{Re}(\beta\bar{\beta}^*) \quad (7a)$$

$$\Pr_S(e\bar{e}\,|\,c\bar{c}) = |\gamma+\bar{\gamma}|^2 = |\gamma|^2 + |\bar{\gamma}|^2 + 2\,\mathrm{Re}(\gamma\bar{\gamma}^*) \quad (7b)$$

Similarly in the antisymmetric case,

$$\Pr_A(d\bar{d}\,|\,c\bar{c}) = |\beta-\bar{\beta}|^2 = |\beta|^2 + |\bar{\beta}|^2 - 2\,\mathrm{Re}(\beta\bar{\beta}) \quad (7c)$$

$$\Pr_A(e\bar{e}\,|\,c\bar{c}) = |\gamma-\bar{\gamma}|^2 = |\gamma|^2 + |\bar{\gamma}|^2 - 2\,\mathrm{Re}(\gamma\bar{\gamma}) \quad (7d)$$

Note here the difference between these formulas and the distinguishable case. Thus $\Pr_S(d\bar{d}\,|\,c\bar{c})$ is the probability of one particle emerging at d and the other at $\bar{d}$, so it is really the counterpart of (6b), not of the expression in (6a). But then the two expressions are still not the same, at least if $\mathrm{Re}(\beta\bar{\beta}^*)$ is nonzero. Consequently, the presence of the Re(...) on the rhs sides of ~~(7d.~~ (7a → d) are indicators of "quantum interference"

Note that because of the unitarity condition (4h) it is the case that, as expected,

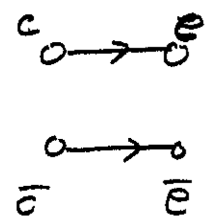
$$Pr_S(d\bar{d} \mid c\bar{c}) + Pr_S(e\bar{e} \mid c\bar{c}) = 1,$$

as one would certainly expect: the particles have to go somewhere. And of course

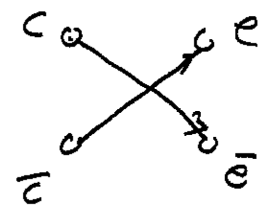
$$Pr_A(d\bar{d} \mid c\bar{c}) + Pr_A(e\bar{e} \mid c\bar{c}) = 1$$

Creation and annihilation operators

Consider the situation in which two particles are initially in the $c, \bar{c}$ locations (sites/orbitals) and one has amplitudes indicated as follows



$= \gamma$



$= \bar{\gamma}$

What do we mean? First take place of AI particles 1 and 2 at $c, \bar{c}$ respectively. Then in one time step

$$T\,|c\bar{c}\rangle = T\,|1c, 2\bar{c}\rangle =$$

$$= \gamma\,|e\bar{e}\rangle + \bar{\gamma}\,|\bar{e}e\rangle + \cdots$$

$$= \gamma\,|1e, 2\bar{e}\rangle + \bar{\gamma}\,|1\bar{e}, 2e\rangle + \cdots$$

where ... indicates "other possibilities" not covered by what is explicitly shown. Since we are dealing with AI particles we also, of course, have

$$T\,|\bar{c}c\rangle = \gamma\,|\bar{e}e\rangle + \bar{\gamma}\,|e\bar{e}\rangle + \cdots$$

Note that γ and $\bar{\gamma}$ may be very different.

Now let us try writing this in field theoretical form using creation and annihilation operators for **bosons**. Then it seems sensible to write

$$T = \gamma\,(b_e^+ b_c)(b_{\bar e}^+ b_{\bar c}) + \bar\gamma\,(b_e^+ b_{\bar c})(b_{\bar e}^+ b_c) + \cdots \qquad (12a)$$

as one can visualize $b_e^+ b_c$ as "removing" a particle that is at c and "creating" a particle at e. Similarly the other terms

If, on the other hand, we were dealing with AI particles so that particles 1 and 2 are initially at $c, \bar c$, or at $\bar c, c$, then we might write instead something like

$$T = \gamma\,(b_{1e}^+ b_{1c})(b_{2\bar e}^+ b_{2\bar c}) + \gamma\,(b_{2e}^+ b_{2c})(b_{1\bar e}^+ b_{1\bar c})$$
$$+ \bar\gamma\left[(b_{1e}^+ b_{1\bar c})(b_{2\bar e}^+ b_{2c}) + (b_{2e}^+ b_{2\bar c})(b_{1\bar e}^+ b_{1c})\right] + \cdots \qquad (12b)$$

in what is a rather awkward but ultimately unambiguous notation which tells us what T does to states ~~$1c\bar c$~~ $|1c, 2\bar c\rangle$ or $|1\bar c, 2c\rangle$

One might use an alternative notation where g, g^+ denotes annihilation/creation operators for particle 1, h, h^+ those for particle 2, and

$$T = \gamma (g^+_e g_c)(h^+_{\bar{e}} h_{\bar{c}}) + \gamma (g^+_{\bar{e}} g_{\bar{c}})(h^+_e h_c)$$

$$+ \bar{\gamma} \left[(g^+_e g_{\bar{c}})(h^+_{\bar{e}} h_c) + (g^+_{\bar{e}} g_c)(h^+_e h_{\bar{c}}) \right] + \cdots$$

Reverting to (12a), and assuming the b, b^+ satisfy the usual commutation relations, we may rewrite it as:

$$T = (\gamma + \bar{\gamma})\,(b_e^+ b_{\bar{e}}^+ \, b_c b_{\bar{c}}) + \cdots$$

which is to say, if we apply T to $|c\bar{c}, S\rangle$ we get

$$T\,|c\bar{c}, S\rangle = \int^{(\gamma+\bar{\gamma})} |e\bar{e}, S\rangle + \cdots$$

Notice that here only the combination of $\gamma + \bar{\gamma}$ occurs. If the entire discussion is about identical bosons, there is no place for a separate γ and $\bar{\gamma}$ — these are individually defined only for A I particles

What should we do in the case of fermions?

If we replace

$$b_j \to f_j \tag{14a}$$

in (12a) the result is

$$\begin{aligned} T &= \gamma (f_e^+ f_c)(f_{\bar{e}}^+ f_{\bar{c}}) + \bar{\gamma}(f_e^+ f_{\bar{c}})(f_{\bar{e}}^+ f_c) + \cdots \\ &= (\bar{\gamma} - \gamma)\, f_e^+ f_{\bar{e}}^+ f_c f_{\bar{c}} + \cdots \\ &= (\gamma - \bar{\gamma})\, f_e^+ f_{\bar{e}}^+ f_{\bar{c}} f_c + \cdots \end{aligned} \tag{14b}$$

where the final result's phase depends on what convention we employ for ordering the orbital.

As a check, try applying (12 b) to the explicitly anti-symmetrized state

$$\begin{aligned} |\psi\rangle &= |c\bar{c}\rangle - |\bar{c}c\rangle = |1c, 2\bar{c}\rangle - |1\bar{c}, 2c\rangle \\ &= \sqrt{2}\,|c\bar{c}, A\rangle \end{aligned} \tag{14c}$$

$$\begin{aligned} T|\psi\rangle &= \gamma |1e, 2\bar{e}\rangle - \gamma |1\bar{e}, 2e\rangle \\ &\quad + \bar{\gamma}|1\bar{e}, 2e\rangle - \bar{\gamma}|1e, 2\bar{e}\rangle + \cdots \\ &= (\gamma - \bar{\gamma})\left[\,|1e, 2\bar{e}\rangle - |1\bar{e}, 2e\rangle\,\right] + \cdots \end{aligned} \tag{14d}$$

which is consistent with (14b) and a convention in which

$$\begin{aligned} |c\bar{c}\rangle_A &= f_c^+ f_{\bar{c}}^+ |\phi\rangle \\ |e\bar{e}\rangle_A &= f_e^+ f_{\bar{e}}^+ |\phi\rangle \end{aligned} \tag{14e}$$

The point is that

$$|e\bar{e}\rangle_A \langle c\bar{c}|_A = f_e^+ f_{\bar{e}}^+ \, |\phi\rangle\langle\phi| \, f_{\bar{c}} f_c$$

which is identical to

$$f_e^+ f_{\bar{e}}^+ f_{\bar{c}} f_c$$

if the latter is applied to a state in which there is precisely one particle in c and one in $\bar{c}$, since in that case the vacuum projector $|\phi\rangle\langle\phi|$ has no (additional) effect.

We can, naturally, use the same sorts of arguments in which the role of $e, \bar{e}$ is played by $d, \bar{d}$ and hence arrive at counterparts of (4c) + (4d), where in the symmetrical case we write

$$T_S = (\beta + \bar{\beta})(b_d^+ b_{\bar{d}}^+ b_c b_{\bar{c}}) + (\gamma + \bar{\gamma}) b_e^+ b_{\bar{e}}^+ b_c b_{\bar{c}} + \cdots$$

and for the antisymmetric case

$$T_A = (\beta - \bar{\beta})(f_d^+ f_{\bar{d}}^+ f_{\bar{c}} f_c) + (\gamma - \bar{\gamma})(f_e^+ f_{\bar{e}}^+ f_{\bar{c}} f_c) + \cdots$$

where in both cases $\cdots$ refers now to what T_S (T_A) does in case we have some other initial state than $|c\bar{c}\rangle_S$ (or $|c\bar{c}\rangle_A$). a

And, of course, if we allow not just one but many scattering possibilities, $d_1, d_2, \ldots$, then those terms will need to be added in.