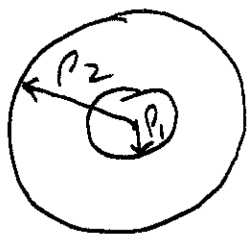
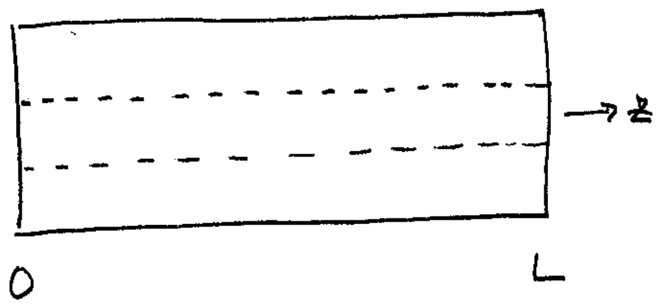
33-756 Quantum Mechanics II

Spring 2011

Notes on Coaxial Resonator

Consider coaxial cable but with no dielectric (in order to make things simple) which is shorted out at $z = 0$ and $z = L$, and in which the inner and outer conductor have radii ρ_1 and ρ_2





we use cylindrical coordinates (r, ϕ, z)

The idea is to consider a mode in which the electric field is always radially outward with no ϕ or z components, and which furthermore vanishes at $z = 0$ and $z = L$ corresponding to the metal plates that "short out" the coax and turn it into a resonator.

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We assume that the entire setup can be described using a vector potential $\vec{A}(\vec{r},t)$ through which the electric and magnetic fields are given (SI units) as

$$\vec{E}(\vec{r},t) = -\partial \vec{A}/\partial t \qquad (2a)$$

$$\vec{B}(\vec{r},t) = \nabla \times \vec{A} \qquad (2b)$$

Let us then write

$$\vec{A} = b(t)\vec{F}(\vec{r}) \qquad (2c)$$

~~where $\vec{A}_0$~~

where in the region between inner and outer conductors,

$$\vec{F} = (F_0/r)\sin kz \;\hat{r} \qquad (2d)$$

with $\hat{r}$ a unit vector in the radial direction,

$$k = n\pi/L \qquad (2e)$$

For cylindrical coordinates, see Physicist's Desk Ref (Physics Vade Mecum 2d) p. 15

$$(\nabla\times\vec{A})_r = \frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}$$

$$(\nabla\times\vec{A})_\phi = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$(\nabla\times\vec{A})_z = \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial\phi}$$

$$\mathrm{div}\cdot A = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

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$$(\nabla^2 \vec{A})_r = \nabla^2 A_r - \frac{2}{r^2}\frac{\partial A_\phi}{\partial r} - \frac{A_r}{r^2}$$

$$(\nabla^2 \vec{A})_\phi = \nabla^2 A_\phi + \frac{2}{r^2}\frac{\partial A_r}{\partial \phi} - \frac{A_\phi}{r^2}$$

$$(\nabla^2 \vec{A})_z = \nabla^2 A_z$$

Here the first ∇^2 is that of a scalar, and in cylindrical coordinates:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

In the present case because

$$F_r = (F_0/r)\sin kz \qquad F_\phi = F_z = 0$$

we find

$$\operatorname{div}\vec{F} = 0$$

$$(\nabla\times\vec{F})_r = (\nabla\times\vec{F})_z = 0$$

$$(\nabla\times\vec{F})_\phi = \frac{\partial F_r}{\partial z} = (kF_0/r)\cos kz$$

Therefore the electric and magnetic fields are

$$E_r = -\dot{b}\,F_r = -\dot{b}(F_0/r)\sin kz$$

$$B_\phi = (b\,k\,F_0/r)\cos kz$$

This is as expected: $\vec{E}$ is purely radial, $\vec{B}$ is purely azimuthal. $\vec{E}$ vanishes at $z = 0$ or L

Check: We want

$$\nabla\times\vec{E} = -\partial\vec{B}/\partial t$$

and

$$(\nabla\times\vec{E})_\phi = -(\dot{b}kF_0/r)\cos kz$$

$$(\partial\vec{B}/\partial t)_\phi = (\dot{b}kF_0/r)\cos kz$$

Next we invoke the fact that $\vec{A}$ satisfies the wave equation

$$\frac{\partial^2 \vec{A}}{\partial t^2} = c^2 \nabla^2 \vec{A}$$

or

$$\ddot{b}\,\vec{F} = b\,c^2 \nabla^2 \vec{F}$$

But $\vec{F}$ has only a radial component, so, ref. 2 that does not depend on Φ, so

$$(\nabla^2 \vec{F})_\phi = (\nabla^2 \vec{F})_z = 0$$

$$\cancel{(\nabla^2 \vec{F})_r = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial F_0}{}\right)}$$

$$(\nabla^2 \vec{F})_r = F_0 \sin kz \underbrace{\left\{ \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\frac{1}{r}\right) - \frac{1}{r^3}\right\}}_{0}$$

$$-k^2 F_0 \sin kz / r$$

Thus we get is $\ddot{b} = -bc^2k^2$

which means that it is periodic ~~tolt~~ motion with angular frequency

$$\omega = ck = n\pi c/L$$

as expected, where $n = 1, 2, 3 \ldots$

Energy is given by (H denotes Hamiltonian)

$$H = \epsilon_0 \int d^3\vec{r} \left[\vec{E}^2 + c^2 \vec{B}^2 \right]$$

Check: $\epsilon_0 \vec{E}^2 = \vec{E} \cdot \vec{D}$ with $\vec{D} = \epsilon_0 \vec{E}$

$$c^2 \epsilon_0 \vec{B}^2 = \vec{B} \cdot \vec{H}$$

since $c^2 = 1/\epsilon_0 \mu_0$ and $\vec{B} = \mu_0 \vec{H}$

Only the components E_r and B_ϕ are nonzero, and they are given by

$$E_r = -\dot{b}\,(F_0/r) \sin kz$$

$$B_\phi = bk\,(F_0/r) \cos kz$$

Therefore

$$cB_\phi = b\omega\,(F_0/r) \cos kz$$

$$E_r^2 + c^2 B_\phi^2 = \vec{E}^2 + c^2 \vec{B}^2 = (F_0/r)^2 \left[\dot{b}^2 \sin^2 kz + b^2 \omega^2 \cos^2 kz \right]$$

Note that if we write

$$b = b_0 \cos \omega t$$

then at $t=0$ all the energy is in the magnetic field; at $t = T/4$ it is all in the electric field, and in terms of spatial location switches back and forth from ends to center of cavity

Carrying out the integral over the cavity volume:

$$\int d^3\vec{r} = 2\pi \int dz \int_{\rho_1}^{\rho_2} r\,dr \qquad (6a)$$

$$\int_0^L dz \sin^2 kz = \int_0^L dz \cos^2 kz = L/2 \qquad (6b)$$

$$\int_{\rho_1}^{\rho_2} (F_0/r)^2 \, r\,dr = F_0^2 \int_{\rho_1}^{\rho_2} dr/r = F_0^2 \ln(\rho_2/\rho_1) \qquad (6c)$$

Therefore

$$H = \epsilon_0 F_0^2 (\pi L) \ln\left(\frac{\rho_2}{\rho_1}\right) \left[\dot{b}^2 + b^2\omega^2\right] \qquad (6d)$$

Check on dimensions.

Electric field $E \sim \dot{b} F_0 / l$

So $\dot{b}^2 F_0^2 \sim E^2 \times l^2$. $\epsilon_0 E^2 \sim$ Energy/volume.

So dimensions of H are O.K.

Let us rewrite (6d) in the form

$$H = H_0 \left[\dot{b}^2 + b^2\omega^2\right]$$

$$H_0 = (\pi L)\epsilon_0 F_0^2 \ln(\rho_2/\rho_1)$$

In order to obtain a more intuitive understanding of the energy in (6d) let us assume that

$$E_0 = \dot{b}\,(F_0/\rho_1) \qquad (7a)$$

is the maximum value in space and time of the electric field: this occurs at $z = L/2$, at the midpoint of the cavity, and at $r = \rho_1$, the surface of the inner conductor.

Set

$$b = \sin\omega t \qquad (7b)$$
$$\dot{b} = \omega\cos\omega t$$

so that

$$\dot{b}^2 + b^2\omega^2 = \omega^2 \qquad (7c)$$

Solve (7a) when $t=0$ to get

$$F_0 = E_0\,\rho_1/\omega \qquad (7d)$$

and insert this in (6d) to get

$$H = \epsilon_0\,(\rho_1/\omega)^2 \cdot \pi L \cdot \ln(\rho_2/\rho_1) \cdot \omega^2 \qquad (7e)$$

or

$$H = (\epsilon_0 E_0^2)\,(\pi L \rho_1^2)\,\ln(\rho_2/\rho_1) \qquad (7f)$$

where one recognizes $\pi L \rho_1^2$ as the volume occupied by the inner conductor. As $\epsilon_0 E_0^2$ is energy per unit volume in S.I., dimensions are correct

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Note that (7f) applies for any of the modes of the form under consideration, i.e., for any k given by (2e), as it involves neither k nor the frequency

See .21+ for specific magnitudes.

Quantization of mode

Let us assume that we have chosen F_0 so that $b(t)$ is dimensionless, which is to say F_0 has dimensions of length $\times \vec{F}$ = length $\times \vec{A}$ = length $\times$ time $\times \vec{E}$

Then it will be convenient to rewrite (6d) in the form

$$H = H_0 \left[\dot{b}^2 + b^2 \omega^2 \right]$$

in which the quantity

$$H_0 = \epsilon_0 F_0^2 [\pi L] \ln(\rho_2/\rho_1)$$

has dimensions of energy $\times$ (times)^2. We play the usual trick and write

$$\frac{H}{\hbar\omega} = \frac{H_0}{\hbar\omega} \dot{b}^2 + \frac{H_0 \omega}{\hbar} b^2$$

We then ~~suppose~~ introduce a dimensionless amplitude, write it as

$$\beta = \left(\frac{H_0 \omega}{2\hbar}\right)^{1/2} b = \frac{1}{\sqrt{2}}(a + a^+)$$

and a dimensionless velocity

$$\gamma = \left(\frac{H_0 \omega}{2\hbar}\right)^{1/2} \dot{b} = \frac{-i}{\sqrt{2}}(a - a^+)$$

Then

$$\frac{H}{\hbar\omega} = \frac{1}{2}(\beta^2 + \gamma^2) = \frac{a a^+ + a^+ a}{2}$$

We can then write for a combination of many modes labeled by n in $k = n\pi/L$, (2c), the energy, etc., as a ~~contin~~ sum

$$H = \sum_k \hbar \omega_k \left(a_k^{+} a_k + 1/2 \right)$$