# 33-756 Quantum Mechanics II

Spring 2011

Notes on Quantization of Stretched String

In this section I discuss a stretched string with fixed end points in terms of its normal modes. Things are worked out in some detail for the classical modes, including orthogonality relations for the mode functions. "Quantization" ~~conditions are~~ is then carried out in the standard fashion, and the string displacement and conjugate momentum operators are expressed in terms of normal mode creation and annihilation operators. (This generalizes to a simple case the discussion of coupled oscillators in Sec. 4)

Consider the example of a <u>stretched string</u>

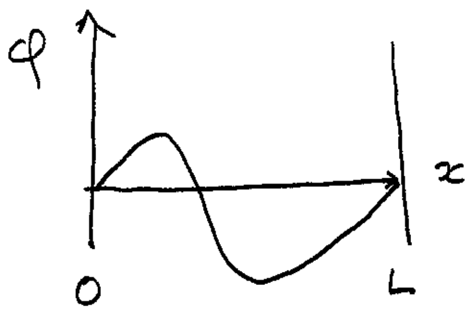
for which

$T$ = tension (2a)

$\mu$ = mass per unit length (2b)

$\varphi(x,t)$ = transverse displacement (2c)

of point located at $x$ (as measured along the string) as a function of time.



(2d)

assume string of length L which is <u>fixed at both ends</u>

Then the kinetic energy and potential energy are given by

$$T = \frac{\mu}{2}\int_0^L \dot{\varphi}^2 dx = \frac{\mu}{2}\int_0^L \left(\frac{\partial\varphi}{\partial t}\right)^2 dx \qquad (2e)$$

$$V = \frac{\mu c^2}{2}\int_0^L \left(\frac{\partial\varphi}{\partial x}\right)^2 dx = \frac{\tau}{2}\int_0^L \left(\frac{\partial\varphi}{\partial x}\right)^2 dx \qquad (2f)$$

where

$$c = \sqrt{\tau/\mu} \qquad (2g)$$

is the speed of wave propagation

Boundary conditions:

$$\varphi(0,t) = 0 = \varphi(L,t) \qquad (2h)$$

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The Lagrangian (use $\mathcal{L}$ so as not to confuse with the length of string) is

$$\mathcal{L} = T - V = \frac{1}{2}\int_0^L \left[\mu\left(\frac{\partial\varphi}{\partial t}\right)^2 - T\left(\frac{\partial\varphi}{\partial x}\right)^2\right]dx. \quad (3a)$$

whereas the Hamiltonian is

$$H = T + V \quad (3b)$$

The wave equation is of the form

$$c^2\frac{\partial^2\varphi}{\partial x^2} = \frac{\partial^2\varphi}{\partial t^2} \quad (3c)$$

Suppose one finds a solution to this, ~~call it $\varphi_k$~~ of the form

$$\varphi(x,t) = b(t)f(x) \quad (3d)$$

where

$$b(t) \propto \cos\omega t \text{ or } \sin\omega t. \quad (3e)$$

Then one has the eigenvalue problem

$$c^2\frac{d^2f}{dx^2} = -\omega^2 f \quad (3f)$$

which can be studied relative to a given set of boundary conditions, viz. (2h)

One can introduce a momentum density

by writing

$$\pi(x,t) = \pi = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}} = \mu \dot{\varphi} = \mu \frac{\partial \varphi}{\partial t}$$

As $\mu$ is mass per unit length, $\pi$ has dimensions of momentum per unit length.

One can then write

$$T = \frac{1}{2\mu} \int_0^L \pi(x,t)^2 \, dx$$

The situation with the string is in many ways analogous to the <u>coupled</u> <u>oscillator</u> problem as discussed in Sec. 4, and the analogies help guide how one thinks about the continuous case.

⊙ The kinetic energy (2e) is a sum of "diagonal" terms, and corresponds to the equal mass case, so we the situation contemplated in § 4c

⊙ The potential energy (2f) is not diagonal, and hence the basic intent of a "normal modes" form is to "diagonalize" the potential energy while rewriting the kinetic energy in normal mode "diagonal" form, i.e., maintaining it in a simple form in the normal mode coordinates

⊙ The diagonalization process just discussed can be regarded as finding solutions to an <u>eigenvalue</u> <u>problem</u> associated with a particular <u>symmetrical</u> <u>operator</u>.

• The symmetrical operator will be associated with a <u>real</u> <u>symmetrical</u> <u>inner</u> product, which it is then useful to identify, and is associated with appropriate <u>boundary</u> <u>conditions</u>.

• With appropriate normalization one can associate the eigenstates with "columns" of the analog of a real, ~~symmetrical~~ orthogonal matrix.

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⊙ The normal mode transformation leads to a "unitary" or "real orthogonal" or "Fourier transform" map that relates the original coordinates and momenta / velocities to those of the normal modes

- It is useful to identify this
- The unitary (orthogonality) of such transformations can be put to good use in the algebraic manipulations that take place (to produce "delta functions")

For the problem as formulated on. 2 + with the eval problem for the function $f(x)$ as in (3f), one knows the answer, viz, with $\bar{c}_n > 0$

$$f_n(x) = \bar{c}_n \sin(n\pi x/L) \qquad n = 1, 2, \cdots \qquad (11a)$$

since these functions satisfy the boundary conditions.

$$f(0) = f(L) = 0 \qquad (11b)$$

The corresponding inner product* is

$$\langle f, g \rangle = \int_0^L f(x) g(x) dx$$

and relative to this choice consider the operators

$D = d/dx$, $D^2 = d^2/dx^2$

~~is symmetric. To see this we note that~~ Note that:

$$\langle f, d/dx\, g \rangle = \int_0^L f(x) \frac{dg}{dx} dx = f(x) g(x) \Big|_{x=0}^{x=L} - \int_0^L (df/dx) g(x) dx$$

or $\langle f, Dg \rangle = -\langle Df, g \rangle$

where the boundary terms vanish by the boundary conditions (11b), therefore, ~~and thus, in a slightly non-Diracian notation~~

$$\langle f, d^2g/dx \rangle = + \langle d^2f/dx, g \rangle$$

or $\langle f, D^2 g \rangle = \langle D^2 f, g \rangle$

and hence $d^2/dx^2$ is a symmetric operator as desired.

* Use $\langle , \rangle$ to avoid confusion with later $\langle \mid \rangle$

In addition one notes that

$$\langle f \,|\, d^2g/dx^2 \rangle = \int_0^L f(x)\frac{d^2g}{dx^2}\,dx = f(x)\frac{dg}{dx}\Big|_{x=0}^{x=L}$$

$$-\int_0^L \frac{df}{dx}\frac{dg}{dx}\,dx = -\int_0^L \frac{df}{dx}\frac{dg}{dx}\,dx.$$

or $\langle f, D^2 g\rangle = -\langle Df, Dg\rangle$

But then $-d^2/dx^2 = -D^2$

is a positive operator, since

$$\langle Df, Df\rangle = \langle f, -\frac{d^2}{dx^2} f\rangle = \int_0^L \left(\frac{df}{dx}\right)^2 dx \geq 0$$

This, of course, is what one wants in order to guarantee that the evals of $d^2/dx^2$, which are $-\omega^2$, are negative.

We can use the specified inner product in order to normalize the $f_n(x)$, and write them out explicitly in the form

$$f_n(x) = \sqrt{2/L}\ \sin(n\pi x/L) \qquad (13a)$$

The virtue of using normalized wave functions is that we then expect to find

$$\int_0^L f_m(x)\, f_n(x)\, dx = \delta_{mn} \qquad (13b)$$

as is, indeed, the case, though checking it by using trig functions of angle sums is a bit of a mess: one should write

$$\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right] \qquad (13c)$$

and check that

$$\int_0^L \cos(n\pi x/L) = \frac{L}{n\pi} \sin\frac{n\pi x}{L}\Bigg|_{x=0}^{x=L} = 0 \qquad (13d)$$

The implication of (13b) is that

$$\sum_n f_n(x)\, f_n(x') = \delta(x-x') \qquad (13e)$$

This is true because for any $g(x)$

$$g(x) = \sum_n f_n(x) \int_0^L f_n(x')\, g(x')\, dx' \qquad (13f)$$

Thus multiply (13e) on both sides by $g(x')$ and integrate over $x'$: one arrives at (13f)

One sees at once that (13a) satisfies the eigenvalue equation (13f)

$$\frac{d^2 f_n}{dx^2} = -\left(\frac{n\pi}{L}\right)^2 f_n = -\frac{\omega_n^2}{c^2} f_n \qquad (14a)$$

which means

$$\omega_n = n\pi c/L \qquad (14b)$$

c = speed of wave propagation, We might also write

$$\omega_k = ck \qquad (14c)$$

where

$$k = n\pi/L \qquad (14d)$$

is thot of as a positive real number taking on certain discrete values; of course there is danger of confusion between (14b) and (14c)

We can now rewrite the kinetic and potential energies in terms of the normal modes in the following fashion. We assume that

$$\varphi(x,t) = \sum_n b_n(t) f_n(x) = \sum_n b_n f_n \qquad (15a)$$

Then, see also (16b) for $p_n$:

$$\frac{1}{\mu}\pi(x,t) = \frac{\partial \varphi}{\partial t} = \sum_n \dot{b}_n f_n = \frac{1}{\mu}\sum_n p_n f_n \qquad (15b)$$

$$\frac{\partial \varphi}{\partial x} = \sum_n b_n D f_n$$

where $D$ stands for the operator

$$D = d/dx$$

Therefore we have

$$T = \frac{\mu}{2}\int_0^L \dot{\varphi}^2 dx = \frac{\mu}{2}\langle \dot{\varphi}, \dot{\varphi}\rangle$$

$$= \frac{\mu}{2}\sum_{m,n} \dot{b}_m \dot{b}_n \langle f_m, f_n\rangle = \frac{\mu}{2}\sum_m \dot{b}_m^2$$

$$V = \frac{T}{2}\int_0^L \left(\frac{\partial \varphi}{\partial x}\right)^2 dx = \frac{T}{2}\sum_{m,n} b_m b_n \langle D f_m, D f_n\rangle$$

But $\langle D f_m, D f_n\rangle = -\langle f_m, D^2 f_n\rangle = \frac{\omega_n^2}{c^2}\langle f_m, f_n\rangle$

$$= \frac{\omega_n^2}{c^2}\delta_{mn}$$

Therefore

$$V = \frac{T}{2}\sum_m \frac{\omega_m^2}{c^2} b_m^2 = \frac{\mu}{2}\sum_m \omega_m^2 b_m^2$$

<u>In summary:</u> Given $\varphi(x,t)$ in the form (15a), we have

$$T = \frac{\mu}{2}\sum_n \dot{b}_n^2 \qquad V = \frac{\mu}{2}\sum_n \omega_n^2 b_n^2 \qquad (16a)$$

Then to introduce momentum for the n'th mode, write it as

$$p_n = \frac{\partial T}{\partial \dot{b}_n} = \mu \dot{b}_n \qquad (16b)$$

and hence in a Hamiltonian formulation

$$H = T+V = \frac{1}{2\mu}\sum_n p_n^2 + \frac{\mu}{2}\sum_n \omega_n^2 b_n^2 \qquad (16c)$$

Note on <u>dimensions</u>. One thinks of $\varphi$ as having dimensions of length (transverse displacement of string), which means $\dot{\varphi} \sim l t^{-1}$ $\partial\varphi/\partial x \sim 1$. But $f_n$ has dimension $l^{-1/2}$, see (13a), which means that $b_n \sim l^{3/2}$, see (15a). Now $\mu \sim m l^{-1}$ (mass per unit length). Thus $\mu \dot{b}_n^2 \sim (m/l)(l^{3/2}t^{-1})^2 = m l^2 t^{-2}$ is an energy. $p_n \sim (m/l)(l^{3/2}t^{-1}) \sim m l^{1/2} t^{-1}$ is "momentum/$\sqrt{\text{length}}$", but then $p_n^2/\mu \sim (m^2 l t^{-2})/(m l^{-1}) = m l^2 t^{-2}$ = energy

In order to "quantize" the string, we note that in (16c) we have the energy expressed as a sum of contributions from different modes. So we should start off with a <u>single mode</u>, and for simplicity drop the mode label $n$, and look at

$$H = \frac{1}{2\mu} p^2 + \frac{\mu}{2} \omega^2 b^2 \qquad (21a)$$

which has dimensions of energy [see .16] despite the fact that $p$ and $b$ are a bit peculiar. So if we write

$$\frac{H}{\hbar\omega} = \frac{p^2}{2\mu\hbar\omega} + \frac{\mu\omega}{2\hbar} b^2 \qquad (21b)$$

this quantity will be dimensionless, and therefore ~~in dimensionless~~ it makes sense to write it in terms of dimensionless quantities [cf. 2c.2]

$$\frac{H}{\hbar\omega} = \frac{1}{2} P^2 + \frac{1}{2} Q^2 \qquad (21c)$$

$$P = \frac{p}{\sqrt{\mu\hbar\omega}} \qquad Q = \sqrt{\frac{\mu\omega}{\hbar}}\, b \qquad (21d)$$

$$p = \sqrt{\mu\hbar\omega}\, P \qquad b = \sqrt{\frac{\hbar}{\mu\omega}}\, Q \qquad (21e)$$

The quantization step is then to "promote" P and Q to Hermitian operators which satisfy

$$[P, Q] = -i\ I$$

or to write them in terms of creation and annihilation operators

$$Q = \frac{1}{\sqrt{2}}(a + a^+) \qquad P = \frac{-i}{\sqrt{2}}(a - a^+)$$

$$a = \frac{1}{\sqrt{2}}(Q + iP) \qquad a^+ = \frac{1}{\sqrt{2}}(Q - iP)$$

where

$$[a, a^+] = I$$

Consequently, the mode "variables" $p$ and $b$ are promoted to noncommuting operators

$$p = -i\sqrt{\frac{\mu\hbar\omega}{2}}\ (a - a^+) \qquad b = \sqrt{\frac{\hbar}{2\mu\omega}}\ (a + a^+)$$

and the Hamiltonian to an operator

$$H = \tfrac{1}{2}\hbar\omega\ (aa^+ + a^+a) = \hbar\omega\,(a^+a + \tfrac{1}{2})$$

Each individual mode, as labeled by $n$, can be quantized in exactly the same way; just attach subscript '$n$' to $b, p, P, Q, a, a^+$.

In particular we now have

$$H = \frac{1}{2\mu}\sum_n p_n^2 + \frac{\mu}{2}\sum_n \omega_n^2 b_n^2 = \sum_n \hbar\omega_n\left(a_n^+ a_n + \tfrac{1}{2}\right) \quad (23a)$$

$$p_n = -i\sqrt{\frac{\mu\hbar\omega_n}{2}}\left(a_n - a_n^+\right) \quad (23b)$$

$$b_n = \sqrt{\frac{\hbar}{2\mu\omega_n}}\left(a_n + a_n^+\right) \quad (23c)$$

Thus $p_n$ and $b_n$ are now <u>operators</u> on a suitable Hilbert space

Similarly

$$\varphi(x) = \sum_n b_n f_n(x) \quad (23d)$$

$$\pi(x) = \sum_n p_n f_n(x) \quad (23e)$$

are now <u>operators</u>; note that $f_n(x)$ are simply <u>real</u> numbers, also $x$ is a <u>real</u> number, and therefore $\varphi(x)$, $\pi(x)$ are collections of operators which as functions of $x$ are <u>defined</u> by (23d,e)

Commutators.

We assume the quantum modes are independent, and therefore

$$[a_m, a_n] = 0 = [a_m^\dagger, a_n^\dagger]$$

$$[a_m, a_n^\dagger] = \delta_{mn} I$$

One can then work backwards using (23b,c) to show that

$$[b_m, b_n] = 0 = [p_m, p_n]$$

Note that these are "obviously" true if $m \neq n$, and also obvious for $m = n$ since an operator always commutes with itself. Then

$$[b_m, p_n] = i\hbar \delta_{mn}$$

Inserting this result in (23d,e) one then has

$$[\varphi(x), \pi(x')] = \sum_{m,n} f_m(x) f_n(x') [b_m, p_n]$$

$$= i\hbar \sum_n f_n(x) f_n(x') = i\hbar \delta(x - x')$$

~~so~~ where the last is a consequence of (13e)

$$[\varphi(x), \pi(x')] = i\hbar \delta(x - x')$$