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Creation and Annihilation Operators

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1 Introduction

⊙ Creation and annihilation operators are used in many-body quantum physics because they provide a less awkward notation than symmetrized or antisymmetrized wave functions, and a convenient language for perturbation theory, etc. These notes are not intended to give anything but an introduction. For a much more extended discussion see books on many-body theory, such as Fetter and Walecka, *Quantum Theory of Many-Particle Systems*.

2 Harmonic Oscillators

⊙ A harmonic oscillator is a good place to begin. The creation and annihilation operators satisfy $[a, a^\dagger] = I$, where I (sometimes written as 1) is the identity operator on the corresponding Hilbert space of a single oscillator. If one has r oscillators with a total Hilbert space

$$\bar{\mathcal{H}} = \bar{\mathcal{H}}_1 \otimes \bar{\mathcal{H}}_2 \otimes \cdots \bar{\mathcal{H}}_r \tag{1}$$

there are operators a_j and $a_j^\dagger$ acting on $\bar{\mathcal{H}}_j$, and extended to the entire Hilbert space $\bar{\mathcal{H}}$ in the usual way (tensored with appropriate identity operators) satisfying commutation relations for j and k in the range of 1 to r:

$$[a_j, a_k^\dagger] = \delta_{jk} I, \quad [a_j, a_k] = 0, \quad [a_j^\dagger, a_k^\dagger] = 0, \tag{2}$$

The third equality is a consequence of the second.

• Let

$$|\emptyset\rangle = |0\rangle \otimes |0\rangle \otimes \cdots |0\rangle \tag{3}$$

denote the ground state, which we shall hereafter refer to as the "vacuum." Then a basis for $\bar{\mathcal{H}}$ can be constructed using linear combinations of states of the form

$$\sqrt{n_1! n_2! \cdots n_r!}\; |n_1, n_2, \ldots n_r\rangle = (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \cdots (a_r^\dagger)^{n_r} |\emptyset\rangle, \tag{4}$$

with oscillator j in the state n_j, n_j any nonnegative integer. We shall say that this is a state in which n_j *phonons* are in the j'th *orbital*, with a total of $N = \sum_j n_j$ phonons present in the many-phonon system.

◦ In quantum optics one would say that there are n_j *photons* present in the j'th *mode*.

3 Identical Bosons

⊙ Start with a collection $\{|\alpha_j\rangle\}$, $j = 1, 2, \ldots$, of *single-particle* states or *orbitals*, which form an orthonormal basis of the Hilbert space $\mathcal{F}$ of a single boson of the type under consideration. For each orbital define a corresponding *creation* operator $b_j^\dagger$ and *destruction* operator b_j, and suppose that this collection of operators satisfy the set of commutation relations

$$[b_j, b_k^\dagger] = \delta_{jk} I, \quad [b_j, b_k] = 0, \quad [b_j^\dagger, b_k^\dagger] = 0, \tag{5}$$

the same as (2) when a is replaced with b.

⊙ The b_j and $b_j^\dagger$ are operators acting on a Hilbert space known as *Fock* space, for which we shall now construct a basis.

• Begin with the vacuum $|\emptyset\rangle$ corresponding to no particles present: imagine an empty box. It spans a one-dimensional subspace $\mathcal{H}_0^S$ of the Fock space, and is annihilated by every one of the annihilation operators:

$$b_j|\emptyset\rangle = 0. \tag{6}$$

• The one particle states are of the form

$$|\alpha_j\rangle = b_j^\dagger|\emptyset\rangle \tag{7}$$

and they span a subspace $\mathcal{H}_1^S$ of the Fock space.

• The two particle states that span the subspace $\mathcal{H}_2^S$ are of the form

$$b_j^\dagger b_k^\dagger|\emptyset\rangle = b_k^\dagger b_j^\dagger|\emptyset\rangle \tag{8}$$

where j and k range over all values corresponding to the different orbitals, though to have a set of linearly independent states one needs a restriction, say $j \leq k$. These states are normalized except in the case $j = k$, two particles in the same orbital, in which case $(b_j^\dagger)^2|\emptyset\rangle/\sqrt{2}$ is a normalized state.

• Similarly $b_j^\dagger b_k^\dagger b_l^\dagger|\emptyset\rangle$ are states of three particles, and now it is obvious how to produce states with any number of particles. Just as for two particles, the *order* in which the creation operators are applied makes no difference; they commute with each other, see (5). What distinguishes different basis states is how many particles are present in each orbital. Thus the essential information identifying the different states forming the basis is in the occupation numbers, and in analogy with (4) one can write

$$|n_1, n_2, \ldots\rangle = (b_1^\dagger)^{n_1}(b_2^\dagger)^{n_2}\cdots|\emptyset\rangle/\sqrt{n_1!n_2!\cdots}. \tag{9}$$

⊙ The Fock space itself is defined to be the direct sum of the subspaces containing 0, 1, 2, etc. particles:

$$\mathcal{H}_F^S = \mathcal{H}_0^S \oplus \mathcal{H}_1^S \oplus \mathcal{H}_2^S \oplus \cdots, \tag{10}$$

Note that states corresponding to different numbers of particles are orthogonal to each other. E.g., any state in the two-particle subspace $\mathcal{H}_2^S$ is orthogonal to any state in $\mathcal{H}_1^S$.

∘ Obviously, $\mathcal{H}_F^S$ can contain linear combinations of states with different numbers of particles. While this may at first seem strange, it is no more "unnatural" than harmonic oscillator states, such as coherent states, that do not contain a definite number of phonons. Allowing the number of particles to vary is convenient in applications of quantum mechanics to situations in which particles can appear and disappear (e.g., photons are absorbed), but the formalism is useful in other situations; e.g., when discussing superconductors or Bose-Einstein condensates.

⊙ The definition of creation operators using (7) obviously depends upon the choice of single-particle states. Nothing said thus far determines what these states must be. An alternative choice, say $\{|\hat{\alpha}_j\rangle\}$, will result in a different collection of operators $\{\hat{b}_j^\dagger\}$. These can be written as linear combinations of the operators $\{b_j^\dagger\}$ making up the previous collection, and if the latter satisfy (5) then the same relations will hold with b everywhere replaced with $\hat{b}$.

⋆ Exercise. Show this.

4 Identical Fermions

⊙ For identical fermions associate creation and annihilation operators $f_j^\dagger$ and f_j with the orbital or single-particle state j, just as in the case of identical bosons, but now but instead of commutators the operators satisfy analogous relations using *anticommutators*

$$\{f_j, f_k^\dagger\} = \delta_{jk} I, \quad \{f_j, f_k\} = 0, \quad \{f_j^\dagger, f_k^\dagger\} = 0. \quad \{A, B\} := AB + BA. \tag{11}$$

⊙ The Fock space is again constructed starting with the vacuum $|\emptyset\rangle$, which is annihilated by all the f_j, and then forming one-particle states

$$|\alpha_j\rangle = f_j^\dagger |\emptyset\rangle \tag{12}$$

which span the one-particle subspace $\mathcal{H}_1^A$. A basis of two particle states is provided by

$$f_j^\dagger f_k^\dagger |\emptyset\rangle = -f_k^\dagger f_j^\dagger |\emptyset\rangle, \tag{13}$$

where in order to avoid overcounting we can require $j < k$. Unlike the case of bosons, $j = k$ does not occur, because (11) tells us that $(f_j^\dagger)^2 = 0$. Thus there are no states with two fermions in the same orbital.

• The state produced by applying $f_j^\dagger f_k^\dagger$ to the vacuum differs from that obtained using $f_k^\dagger f_j^\dagger$ by a minus sign. Thus the two are not linearly independent, and only one should enter in a list of basis states. Two quantum states which differ by an overall phase have the same physical significance. However, keeping track of signs is important if, as is often the case, one is considering various linear combinations (superpositions) of states of two particles.

⋆ Exercise. Show this by constructing some examples. Will $f_1^\dagger f_2^\dagger + f_1^\dagger f_3^\dagger$ applied to $|\emptyset\rangle$ yield the same result as $f_1^\dagger f_2^\dagger + f_3^\dagger f_1^\dagger$? They would be identical if we were dealing with bosons (f replaced with b).

⊙ The Fock space allowing for variable numbers of identical fermions is then the direct sum of a collection of mutually-orthogonal subspaces:

$$\mathcal{H}_F^A = \mathcal{H}_0^A \oplus \mathcal{H}_1^A \oplus \mathcal{H}_2^A \oplus \cdots . \tag{14}$$

⊙ Suppose we have two different species of fermions, e.g., electrons and protons. In that case use the tensor product of the Fock spaces, with the electron operators commuting with the proton operators. Same principle if there are bosons along with the fermions, or several distinct species of bosons.

5 Operators

⊙ In many-body quantum mechanics it is generally convenient to express the operators of interest using creation and annhilation operators. In the following discussion we consider identical bosons, but similar results hold for fermions.

⊙ Consider the Hamiltonian H. For convenience—this is not essential—we assume the one-particle basis states are eigenstates of the Hamiltonian:

$$H|\alpha_j\rangle = \epsilon_j |\alpha_j\rangle. \tag{15}$$

If we then make use of (7) we can write

$$H = \sum_j \epsilon_j |\alpha_j\rangle\langle\alpha_j| = \sum_j \epsilon_j b_j^\dagger |\emptyset\rangle\langle\emptyset| b_j. \tag{16}$$

• Because of the projector $|\emptyset\rangle\langle\emptyset|$ on the right hand side, defining H this way means that it will give zero when applied to any state with two or more particles present. Maybe that is what we want. On the other hand, if instead we write

$$H = \sum_j \epsilon_j b_j^\dagger b_j, \tag{17}$$

then we have something which will give the same results as (16) when applied to states of just one particle, but looks a bit simpler. When applied to states of morre than one particle it will give an energy equal to the sum of the number of particles in each orbital multiplied by the energy of the orbital. Thus the energy of a system of noninteracting particles. To be sure, the particles may interact with each other, in which case we will need to add additional terms to the right side of (17) in order to have the complete Hamiltonian. But it does look as if (17) is a "natural" place to start.

⊙ Operators which can be written in the general form

$$Q = \sum_{j,k} q_{jk} b_j^\dagger b_k, \tag{18}$$

with the q_{jk} some numerical (in general complex) coefficients are called "one body operators." They conserve the number of particles, and one can "visualize" them in terms of moving a particle from orbital k to orbital j. In a similar way, "two body operators" have the general form

$$R = \sum_{j,k} r_{jk;lm} b_j^\dagger b_k^\dagger b_l b_m. \tag{19}$$

• It is not absolutely essential to write the creation operators to the left and the annihilation operators to the right in (17) and (18), but this is customary and convenient for various reasons. E.g., two-body operators yield 0 when applied to any state in $\mathcal{H}_1^S$ where only one particle is present, which seems to make sense.

⊙ Operators for fermions can be written in a similar way, using f in place of b, again with creation operators on the left and annihilation operators on the right. In the case of two-body (and three-body, etc.) operators there can be a sign ambiguity because $f_l f_m = -f_m f_l$, so pay attention.

⋆ Exercise. Consider identical spinless particles in a potential well $0 \leq x \leq a$, in which the energy eigenstates of the form $\sin(j\pi x/a)$, $j = 1, 2, \ldots$ are the one-particle states. Construct the one-body operator that corresponds to a potential energy $v\delta(x - a/2)$, i.e., a Dirac delta placed in the middle of the well. Does it make a difference if the particles are bosons or spinless fermions (i.e., fermions which are all in the same spin state)?

⋆ Exercise. For the situation described in the previous exercise, construct the two-body operator for the situation in which two particles at x and x' interact through a potential $w\delta(x-x')$. Does it make a difference if the particles are bosons or spinless fermions?