33-756 Quantum Mechanics II Spring Semester, 2011
Assignment No. 13 (Revised)
Due Friday, April 22

READING:

Interaction of atoms with electromagnetic fields: the material in Le Bellac, Secs. 14.3.1 to 14.3.3 is found in many textbooks. The treatment in Sakurai Sec. 5.7 puts more emphasis on the physics than does Le Bellac. Spontaneous emission as in Le Bellac's Sec. 14.3.4 is absent from textbooks that do not discuss a quantized electromagnetic field.

Rate equations: Le Bellac, Sec. 5.4

We will *not* be taking up Le Bellac Sec. 14.4 (trapping of atoms)

Two-electron atoms: Le Bellac, Sec. 14.5

TOPICS (tentative):

Mon. April 18. Electromagnetic transitions in atoms. Selections rules

Wed. April 20. Electromagnetic transitions. Photoemission and spontaneous decay

Fri. April 22. Multi-electron atoms.

EXERCISES:

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include comments about the lectures, complaints about the course, etc.

2. Atoms with one valence electron are placed in a magnetic field. It is found that light which is circularly polarized such that the electric field at the position of the atom is of the form

$$\mathcal{E}_x = \mathcal{E}_0 \cos\omega t, \quad \mathcal{E}_y = -\mathcal{E}_0 \sin\omega t, \quad \mathcal{E}_z = 0$$

will be strongly scattered when ω is close to $\omega_{ba} = (E_b - E_a)/\hbar > 0$, whereas very little light of the opposite circular polarization but the same frequency, $\mathcal{E}_y = +\mathcal{E}_0 \sin\omega t$ and $\mathcal{E}_x$ the same as before, is scattered.

a) What is $\Delta m = m_b - m_a$ for this transition? You may answer this question using either a semi-classical picture of what is going on, or by employing first-order time-dependent perturbation theory. (It does no harm to use both.)

b) What will be the polarization of the light scattered out from the atoms in a direction along the x axis? (Hint: It will be linearly polarized.)

c) If the only information you were given was that the scattered light had the polarization you determined in (b), what could you say about Δm for the atomic transition?

3. Based on Cohen-Tannoudji Ch. XIII, Complement E, exercise 7.

Consider a particle of mass m in one dimension with a potential $V(x) = -\alpha\delta(x)$ with $\alpha > 0$. There is a single bound state of energy $E_0 = -m\alpha^2/2\hbar^2$; the corresponding normalized wave function is $\phi_0(x) = \sqrt{m\alpha/\hbar^2}\ \exp[-m\alpha|x|/\hbar^2]$. For each positive energy $E = \hbar^2k^2/2m$ there are two stationary wave functions, one corresponding to an incident particle coming from the left and the other an incident particle coming from the right. The former is given by

$$\chi_k(x) = \begin{cases} \left\{e^{ikx} - e^{-ikx}/[1 + i\hbar^2 k/m\alpha]\right\}/\sqrt{2\pi} & \text{for } x < 0, \\ \left\{[i\hbar^2 k/m\alpha]e^{ikx}/[1 + i\hbar^2 k/m\alpha]\right\}/\sqrt{2\pi} & \text{for } x > 0. \end{cases}$$

a) [Optional] Show that the $\chi_k(x)$ satisfy the orthonormalization relation (in the extended sense):

$$\langle \chi_k | \chi_{k'} \rangle = \delta(k - k').$$

The following relation may be helpful:

$$\int_{-\infty}^{0} e^{iqx}\, dx = \int_{0}^{\infty} e^{-iqx}\, dx = \lim_{\epsilon \to 0+} \frac{1}{\epsilon + iq} = \pi\delta(q) - i\mathcal{P}(1/q),$$

where $\mathcal{P}$ denotes the principal value integral.

b) Calculate the density of states $\rho(E)$ for a positive energy E.

c) Calculate the matrix element $\langle \chi_k | X | \phi_0 \rangle$ of the position operator X between the bound state $|\phi_0\rangle$ and the positive energy state $|\chi_k\rangle$ whose wave function was given above.

d) The particle, assumed to be charged with charge q, interacts with an electric field oscillating at the angular frequency ω. The corresponding perturbation is

$$W(t) = -q\mathcal{E} \sin \omega t,$$

where $\mathcal{E}$ is a constant. Calculate using the golden rule the transition probability w per unit time to an arbitrary positive energy state (the photoelectric or photoionizatin effect). How does w vary with ω and $\mathcal{E}$?

4. Consider a system of n atoms, which can either be in a state a or b, and suppose that there are processes which occasionally cause an atom to make a transition from a to b or vice versa. Let there be n_a atoms in state a, n_b atoms in state b, and suppose the dynamics can be described by rate equations:

$$\frac{dn_a}{dt} = W_{ab}\, n_b - W_{ba}\, n_a = -\frac{dn_b}{dt},$$

where the transition rates W_{ab} and W_{ba} are independent of time.

a) Find the steady state values $\bar{n}_a$ and $\bar{n}_b$ of n_a and n_b, and their ratio $\rho = \bar{n}_b/\bar{n}_a$, in terms of n and the W's. Also look at the transient behavior, assuming that at time $t = 0$, $n_a = n$ and $n_b = 0$.

b) What sort of observations of n_a or n_b as functions of time could be made to determine W_{ab} and W_{ba}?

c) If b is an excited state of an atom, a is the ground state, and the atom is illuminated with light of an appropriate sort with an intensity I, one can write

$$W_{ab} = rI + w_{ab}, \quad W_{ba} = rI,$$

where r is a constant, and w_{ab} is the rate of spontaneous decay of an atom. Sketch the fraction $f_b = \bar{n}_b/n$ of atoms in the excited state under steady-state conditions as a function of I.

d) Suppose that a beam of light of intensity I is illuminating a cell containing n atoms. The rI terms in the rate equations correspond to processes in which (i) a photon of energy ϵ_{ba} is absorbed from the light beam as an atom is excited, and (ii) a photon is returned to the beam as an atom returns to its ground state due to induced emission. The spontaneous emission term w_{ab} results in photons being emitted in all directions, rather than being returned to the original beam of light. Find an expression for the net rate R at which energy is being removed from the beam due to processes (i) and (ii) together, assuming steady state conditions. What happens to R in the limit of large I? Can you give a simple explanation for this based on energy conservation?