33-756 Quantum Mechanics II Spring Semester, 2011
Assignment No. 5
Due Friday, Feb. 11

READING:

Harmonic oscillator: LB Secs. 11.1, 11.2.

TOPICS (tentative):

Mon. Feb. 7. Finish up Wigner-Eckart, including parity. Introduction to harmonic oscillator
Wed. Feb. 9. Harmonic oscillator wave functions and matrix elements
Fri. Feb. 11. Coherent states

EXERCISES:

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include comments about the lectures, complaints about the course, etc.

2. a) The following *recursion* (or *recurrence*) relations for Clebsch-Gordan coefficients are sometimes useful.

$$\begin{aligned}
\text{(i)}\ \sqrt{J(J+1)-M(M+1)}\,\langle j_1,j_2;m_1,m_2|J,M+1\rangle \\
&= \sqrt{j_1(j_1+1)-m_1(m_1-1)}\,\langle j_1,j_2;m_1-1,m_2|J,M\rangle \\
&+ \sqrt{j_2(j_2+1)-m_2(m_2-1)}\,\langle j_1,j_2;m_1,m_2-1|J,M\rangle
\end{aligned}$$

$$\begin{aligned}
\text{(ii)}\ \sqrt{J(J+1)-M(M-1)}\,\langle j_1,j_2;m_1,m_2|J,M-1\rangle \\
&= \sqrt{j_1(j_1+1)-m_1(m_1+1)}\,\langle j_1,j_2;m_1+1,m_2|J,M\rangle \\
&+ \sqrt{j_2(j_2+1)-m_2(m_2+1)}\,\langle j_1,j_2;m_1,m_2+1|J,M\rangle
\end{aligned}$$

Indicate how to derive (i). [Hint: Put $J_+$ inside a matrix element.]

b) Let $R^{(k_1)}_{q_1}$ and $S^{(k_2)}_{q_2}$ be irreducible tensor operators on the same space. Show that

$$T^{(K)}_Q = \sum_{q_1,q_2} \langle k_1,k_2,q_1,q_2|KQ\rangle\, R^{(k_1)}_{q_1}\, S^{(k_2)}_{q_2}$$

satisfies the commutation relations

$$[J_z, T^{(K)}_Q] = Q T^{(K)}_Q, \quad [J_\pm, T^{(K)}_Q] = \sqrt{K(K+1)-Q(Q\pm 1)}\, T^{(K)}_{Q\pm 1}$$

and is therefore an irreducible tensor operator of rank $K$ unless it is zero. Why is it important that it be nonzero? Do not assume that the $R$ and $S$ operators commute. [Hints. $[A,BC] = [A,B]C + B[A,C]$. The recursion relations in (a) might be helpful.]

3. a) Find irreducible tensor operators that span the six-dimensional operator space generated by linear combinations of $x^2$, $y^2$, $z^2$, $xy$, $xz$, $yz$, where $x^2$ means multiplication by $x^2$. Let one of the irreducible operators be $z^2 + (\text{constant})\, r^2$, with an appropriate choice for the constant, where $r^2 = x^2 + y^2 + z^2$. Write the irreducible operators down explicitly, and show—either directly, or by some other argument—that they satisfy the appropriate commutation relations with $\boldsymbol{J} = \boldsymbol{L} = -i\boldsymbol{r} \times \boldsymbol{\nabla}$. [Hint. This operator space contains two irreducible representations. How might they be related to spherical harmonics?]

b) Consider a collection of quantum states

$$\langle \boldsymbol{r} | \psi_m \rangle = Y_{l=1}^{m}(\theta, \phi) f(r).$$

with $m = -1, 0, 1$. Use the Wigner-Eckart theorem to express the matrix elements

$$A(m', m) = \langle \psi_{m'} | x^2 | \psi_m \rangle, \quad B(m', m) = \langle \psi_{m'} | xy | \psi_m \rangle$$

in terms of the constants

$$\alpha = \langle \psi_0 | r^2 | \psi_0 \rangle, \quad \beta = \langle \psi_0 | z^2 | \psi_0 \rangle,$$

and appropriate Clebsch-Gordan coefficients. In each case find the selection rules, i.e., the values of $m$ and $m'$ for which $A(m', m)$ or $B(m', m)$ do not vanish (except "accidentally"). Indicate your reasoning.

4. Find approximate values for the amplitude of oscillation in units of $\xi = \sqrt{\hbar / m\omega}$ and the energy (maximum kinetic energy) in units of $\hbar\omega$ for the following oscillators.

a) A 10 g mass suspended in earth's gravity on a thread of length 10 cm, with an amplitude of oscillation of 0.1 radian.

b) A Be ion (atomic mass 9) suspended in a trap where it oscillates at a frequency ($\omega/2\pi$) of 10 MHz and an amplitude of 1 micron.

5. Show that a particle in a three-dimensional harmonic potential $V(r) = \frac{1}{2} K r^2$ is equivalent to three independent harmonic oscillators having the same frequency. Use this fact to find the four lowest energy levels (ground state and first three excited states), and for each energy level its degeneracy. Since the Hamiltonian is invariant under rotations and reflections one would expect the energy eigenspaces to be representations of $O(3)$. Which irreducible representations are present? What about the folk theorem (that energy eigenspaces are irreducible representations)?