33-756 Quantum Mechanics II Spring Semester, 2011
Assignment No. 4 (Revised)
Due Friday, Feb. 4

READING:

Addition of angular momentum, Clebsch-Gordan coefficients, irreducible tensor operators, Wigner-Eckart theorem. These are taken up in LB Sec. 10.6; Cohen-Tannoudji et al. Ch. X (Complements B_X for Clebsch-Gordan and G_X for irreducible tensor operators); Sakurai, Ch. 3; many other books on quantum mechanics.

Harmonic oscillator: LB Sec. 11.1

TOPICS:

Fri. Jan. 28. Addition of angular momentum: tensor product of representations.
Mon. Jan. 31. Clebsch-Gordan coefficients; operator representations of rotations.
Wed. Feb. 2. Wigner-Eckart theorem.
Fr. Feb. 4. Harmonic oscillator

EXERCISES:

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include comments about the lectures, complaints about the course, etc.

2. Le Bellac 10.7.8, spherical well, parts 1 and 2. In part 2 assume the proton and neutron masses are equal when computing the reduced mass.

3. Consider the tensor product of three spaces corresponding to irreps $j = 1/2$, $j = 1$, and $j = 2$ of $SU(2)$. Which irreducible representations are present and how often? Check that the dimensions of the representations add up to what you expect.

4. Your task is to construct the Clebsch-Gordan coefficients $\langle j_1, j_2; m_1, m_2 | J, M\rangle$ for $j_1 = 1$ and $j_2 = 1/2$, as a 6×6 matrix in which the rows are labeled by m_1, m_2 starting at the top with $m_1 = 1, m_2 = +1/2$ followed by $m_1 = 1, m_2 = -1/2$ followed by $m_1 = 0, m_2 = +1/2$ and so forth. Arrange the columns with the largest value of J and $M = J$ the first column, then let M decrease, and after this let J decrease. You can look it up if you want to, but then complete the steps below.

a) A lot of the elements in your matrix, the C-G coefficients for this case, will be 0 because of a certain rule which is easy to remember, and which you should state. Which ones are left over?

b) Derive the entries for all columns involving the largest J value by starting with $|J, M = J\rangle$ and applying to it $J_- = J_{1-} + J_{2-}$ several times in succession to obtain the different states $|J, M\rangle$ as appropriate sums of the $|j_1, j_2; m_1, m_2\rangle$ states. Remember that in the standard convention the matrix element $\langle J, M - 1|J_-|J, M\rangle$ is positive.

c) It is now time to fill in the nonzero entries for the smaller J value. Explain why using the fact that they are real and that the matrix you are constructing is unitary (i.e., real orthogonal) you can pretty much guess what they are, but you cannot be quite sure of the answer. Well, you also want $\langle J, M - 1|J_-|J, M\rangle$ to be positive, but even that does not *entirely* fix things. (There is a standard convention, which you will find in various sources such as Cohen-Tannoudji et al.)

5. The components of the gradient operator $\vec{\nabla} = (\nabla_x, \nabla_y, \nabla_z) = (\partial/\partial x,\, \partial/\partial y,\, \partial/\partial z)$ span a three-dimensional operator space that forms an operator representation of the rotation group for a single particle, for which the infinitesimal generators are $\{L_x, L_y, L_z\}$ (assume $\hbar = 1$).

a) Show that $\vec{\nabla}$ satisfies the commutation relations appropriate for a vector operator.

b) Show that if $\boldsymbol{\omega}$ represents a rotation of $+\omega$ about the z axis, so that

$$U(\boldsymbol{\omega}) = e^{-i\omega L_z}$$

then this rotation applied to $Q = \nabla_x$, and also to $Q = \nabla_y$, in the sense of

$$Q \to Q' = U(\boldsymbol{\omega}) Q U^\dagger(\boldsymbol{\omega})$$

gives you what you might expect. Use the formula

$$e^B A e^{-B} = A + [B, A] + \frac{1}{2!}[B, [B, A]] + \cdots$$

without worrying about matters of convergence, along with the commutators you found earlier.

c) Check the result in (b) for the case $\omega = \pi/2$, counterclockwise rotation of the x, y plane by $90°$ by using

$$Q' F(\boldsymbol{r}) = \Big(U(\boldsymbol{\omega}) Q U^\dagger(\boldsymbol{\omega})\Big) F(\boldsymbol{r}) = U(\boldsymbol{\omega}) \Big(Q G(\boldsymbol{r})\Big),$$

where $F(\boldsymbol{r}) = x + 2y + 3z$ and $G(\boldsymbol{r}) = U^\dagger(\boldsymbol{\omega}) F(\boldsymbol{r})$. I.e., first rotate the function F appropriately (remember that this means rotating its argument in the opposite sense) to get G, then apply $Q = \nabla_x$ or $Q = \nabla_y$, and then rotate again.