33-756 Quantum Mechanics II Spring Semester, 2011
Assignment No. 3
Due Friday, Jan. 28

TOPICS and suggested READING:

Fri. Jan. 21. Angular momentum operators and irreps of SU(2). LB Secs. 10.1, 10.2
Mon. Jan. 24. Orbital angular momentum. LB Sec. 10.3
Wed. Jan. 26. Particle in central potential; hydrogen atom LB Sec. 10.4
We will *not* take up LB Sec. 10.5, angular distribution in decays
Fri. Jan. 28. Addition of angular momentum (tensor product representations). LB Sec. 10.6
Mon. Jan. 31. Addition of angular momentum (continued).

EXERCISES:

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include comments about the lectures, complaints about the course, etc.

2. Le Bellac 10.7.4 parts 1, 2, and 3; *not* part 4. Part 3 as worded is a boring plug-and-chug exericise, so let us making it more interesting. If we were not given the U matrix, could we have come up with it? Think of it in the following way. The matrices T_x and J_x are simply two different matrices for the same operator, expressed using two different orthonormal bases: one that is conveniently labeled $\{|v\rangle\}$ where $v = x$ or y or z, and the other $\{|m\rangle\}$, where $m = +1$ or 0 or -1. Given two such orthonormal bases the matrix $W_{vm} = \langle v|m\rangle$ is unitary. Can you see a reasonable way to express the $|m\rangle$ kets in terms of the $|v\rangle$ kets so that an appropriate unitary emerges? [Hint: The $l = 1$ spherical harmonics in Sec. 10.3.2 might provide some inspiration.]

3. The Pauli operators σ_x, σ_y, and σ_z are not elements of $SU(2)$, but multiplying each by a constant c makes makes it an element of $SU(2)$. Find c, and then find the subgroup of $SU(2)$ generated by $c\sigma_x$, $c\sigma_y$, and $c\sigma_z$. What is the order of this subgroup? Map these to the corresponding rotations in SO(3) to form a subgroup of SO(3). Discuss what elements of this SO(3) subgroup do in geometrical terms (rotations about this and that axis by some angle).

4. For some reason Le Bellac's discussion of rotation matrices in his Sec. 10.2 does not extend to a full set of Euler angles. (See the somewhat strange footnote at the bottom of p. 314.) Your task is to find an expression for the $D^{(j)}_{m'm}$ matrix in terms of the $d^{(j)}_{m'm}$ matrix for a general rotation written in the form

$$\mathcal{R} = e^{-i\phi J_z} e^{-i\theta J_y} e^{-i\psi J_z}.$$

(continued on the next page)

5. a) Consider functions of the form

$$f(x, y, z) = c_1 x + c_2 y + c_3 z,$$

where c_1, c_2, c_3 are complex numbers thought of as a column vector $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = (c_1, c_2, c_3)^{\mathrm{T}}$. Work out the 3×3 matrices for $L_x = -iy\partial/\partial z + iz\partial/\partial y$, L_y, L_z; also L_+, L_- and $L^2 = L_x^2 + L_y^2 + L_z^2$. Check that $[L_+, L_-]$ has the value you expect. Explain how the matrix for L^2, combined with your knowledge about SO(3), tells you that the representation is irreducible *and* gives you the corresponding value of j.

b) The functions in (a) are not part of the standard Hilbert space $\mathcal{H}$ of square-integrable wave functions, because they cannot be normalized. To remedy this defect one can alter the definition and use

$$f(x, y, z) = \mu(r)\big(c_1 x + c_2 y + c_3 z\big),$$

where $\mu(r)$ is a function of $r = \sqrt{x^2 + y^2 + z^2}$. Argue that this will not change your results in (a). [Hint: What is $L_x \mu(r)$?]

c) Assume that in the "standard" $|lm\rangle$ representation the column vector $(0, 0, 1)^{\mathrm{T}}$ in (a) corresponds to $m = 0$. Compute the other $|lm\rangle$ basis vectors by applying L_+ and L_- to $(0, 0, 1)^{\mathrm{T}}$, taking proper account of normalization and phase, and discuss briefly the connection with the corresponding spherical harmonics.