33-756 Quantum Mechanics II Spring Semester, 2011
Assignment No. 2
Due Friday, Jan. 21

READING:

Topics to be taken up in class on Jan. 19 and 21: the groups O(2), SO(2), O(3), SO(3), SU(2). These can be found in various books on group theory and in internet sources; try and get the general idea without getting lost in sophisticated math.

The irreducible representations of SO(3) and SU(2) are what Le Bellac is constructing in his Secs. 10.1 and 10.2; the connection will be discussed in class. We are a bit behind schedule; the hope is by Jan. 24 to get into the material in Le Bellac Secs. 10.3 and 10.4.

EXERCISES:

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include comments about the lectures, complaints about the course, etc.

2. The group O(2) consists of matrices of the form

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad S(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix},$$

where θ and ϕ are angles in the interval $-\pi < \theta, \phi \leq \pi$. (The interval could just as well be from 0 to 2π, and one could use θ in place of ϕ.) The matrices $R(\theta)$ by themselves are pure rotations and constitute the group SO(2).

a) The reflection $S(\phi)$ must be a reflection about a line in the x, y plane. What is this line? (The angle ϕ is chosen so that $S(0)$ is a reflection in the x axis.)

b) Work out the explicit group multiplication and inverse maps in terms of the parametrizations θ and ϕ. That is, find $R(\theta)R(\theta')$, expressing it as one of the other group elements (R or S type), and likewise $R(\theta)S(\phi)$, $S(\phi)R(\theta)$, $S(\phi)S(\phi')$, $R^{-1}(\theta)$, $S^{-1}(\phi)$.

c) The group SO(2) is abelian and its irreps are one dimensional. So there must be a similarity transformation which brings the matrices $R(\theta)$ to (block) diagonal form. Find it. (The answer is not unique; find something that works.)

d) What are the conjugacy classes of O(2)?

e) What are the irreps of O(2)? Write them down as matrices and provide some explanation. [Hint. Most of them are two dimensional. You can probably make a reasonable guess given that you already know the (one-dimensional) irreps of SO(2).]

3. Consider a quantum particle confined to two dimensions: the x, y plane; polar coordinates are $r = \sqrt{x^2 + y^2}$; $x = r\cos\theta$, $y = r\sin\theta$. The groups of interest in what follows are all subgroups of O(2); i.e., they consist of rotations $R(\theta)$ and/or reflections $S(\theta)$. The problem is to identify $\mathcal{G}$, the symmetry group of the Hamiltonian.

a) Assume that the kinetic energy is the usual $(-\hbar^2/2m)\nabla^2$, invariant under rotations and reflections, and the potential energy is

$$V(x, y) = \kappa r^2$$

corresponding to a harmonic oscillator. What is $\mathcal{G}$? What might be a reasonable guess for the possible dimensions of the energy eigenspaces assuming they are irreps of $\mathcal{G}$?

b) Same question, but now assume that

$$V(x,y) = ax^2 + by^2; \quad 0 < a < b.$$

c) Same question, but now assume that

$$V(x,y) = r^2 f(\theta),$$

where $f(\theta)$ is some positive function which is periodic, $f(\theta+\pi/2) = f(\theta)$. Consider two possibilities: (i) there is a reflection symmetry in the sense of some angle α such that $f(\theta) = f(\alpha - \theta)$; (ii) there is no such reflection symmetry.