33-756 Quantum Mechanics II Spring Semester, 2011
Assignment No. 1. Due Friday, Jan. 14

NOTE: Assignments will be placed on the course web page
http://www.andrew.cmu.edu/course/33-756
In the future assignments will NOT be passed out in class.

READING:
Some book on group theory or which provides an introduction to group theory.
Le Bellac, Secs. 10.1 and 10.2.

EXERCISES:

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include comments about the lectures, complaints about the course, etec.

2. a) Show that matrices of the form

$$M(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

where n is any integer, form a representation of the group of integers under addition. Is it a faithful representation?

b) Show that this representation is reducible by finding a (proper) invariant subspace of dimension one.

c) Show that this representation is not *completely* reducible.

3. The group D_3 consists of the six elements $\{e, r, s = r^2, a, b = ar, c = ar^2\}$ where we think of r as a clockwise rotation of the x, y plane by 120°, and a as a reflection in the y axis. The relations are $r^3 = e$, $a^2 = e$, $arar = e$.

a) Willy Smart has found a representation of $D_3 = S_3$ consisting of the six 3×3 matrices

$$M_1 = \frac{1}{2}\begin{pmatrix} -1-\sqrt{3} & 3 & 3+\sqrt{3} \\ 2\sqrt{3} & -1-\sqrt{3} & -2\sqrt{3} \\ -2\sqrt{3} & 3+\sqrt{3} & 2+2\sqrt{3} \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_3 = \frac{1}{2}\begin{pmatrix} -1-\sqrt{3} & -1 & -1+\sqrt{3} \\ 2 & -1+\sqrt{3} & -2 \\ -2 & -1-\sqrt{3} & 0 \end{pmatrix}, \qquad M_4 = \frac{1}{2}\begin{pmatrix} -1+\sqrt{3} & 3 & 3-\sqrt{3} \\ -2\sqrt{3} & -1+\sqrt{3} & 2\sqrt{3} \\ 2\sqrt{3} & 3-\sqrt{3} & 2-2\sqrt{3} \end{pmatrix},$$

$$M_5 = \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & 2 \\ 2 & -2 & -3 \end{pmatrix}, \qquad M_6 = \frac{1}{2}\begin{pmatrix} -1+\sqrt{3} & -1 & -1-\sqrt{3} \\ 2 & -1-\sqrt{3} & -2 \\ -2 & -1+\sqrt{3} & 0 \end{pmatrix}$$

Unfortunately he doesn't remember which matrix represents which group element, though he is sure M_2 corresponds to the identity e. Sally Wise offers to help him out. "Take a look at the characters, Willy. That and a little thought will at least tell you which matrices are associated with which conjugacy classes." Use this hint to identify which of the M_j correspond to rotations and which to reflections; explain your reasoning.

b) Use the character table for D_3 to figure out which irreps (irreducible representations) are present in Willy's representation.

c) Which M_j matrix corresponds to which group element? The answer is not unique; just find one association that works. A little guesswork is allowed, but then check that it works.

4. The symmetry group D_4 of the square is of order 8 and has two generators: r is a clockwise rotation by 90° and a is a reflection in the x axis. We think of the square as having edges aligned with the x and y axes, with vertices at $(1,1)$, $(1,-1)$, etc., so that r carries $(1,1)$ to $(1,-1)$. The group elements can then be written as

$$e,\ r,\ s=r^2,\ t=r^3,\ a,\ b=ar,\ c=ar^2,\ d=ar^3.$$

a) For each element in the group indicate its geometrical significance as a rotation by some angle (about an axis perpendicular to the x,y plane) or a reflection in some line. E.g., c is a reflection in the y axis.

b) What are the conjugacy classes? Do they make sense in terms of the geometrical characterizations you have given in (a)? How does a knowledge of the conjugacy classes tell you the number of inequivalent irreps?

c) Construct the 2×2 matrices which represent the different group elements as transformations of the x,y plane, denoting them by capital letters: I for the identity, R, C, etc. E.g.,

$$A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Check that the characters agree with an appropriate row in the character table, which you can look up somewhere.