

33-658, 758 Quantum Computation and Information Spring Semester, 2014  
Assignment No. 5 (Not to be turned in!)

ANNOUNCEMENT. There will be an hour exam on Tuesday afternoon, February 18, beginning at 3:00 pm (usual class hour). It is closed book, closed notes, no pocket calculators. Bring a pen or pencil. There will be no quantum information seminar that afternoon.

The exam will be based on the material in Assignments 1 to 4, and in the lectures up to and including Thursday, Feb. 6. However, the latter part of that lecture dealing with fully entangled states and dense coding is *not* included.

Problems 1 below is the first hour exam in a previous course. The others are practice problems. None are to be turned in. Solutions will be provided on Thursday, Feb. 13.

READING associated with recent and current lectures:

- CLRS = Cormen, Leiserson, Rivest, and Stein, Introduction to Algorithms (Library reserve)
- DPV = Dasgupta, Papadimitriou, and Vazirani. Algorithms (Library reserve)
- QCQI = Nielsen and Chuang, Quantum Computation and Quantum Information
- QIT = “Quantum Information Types” Course web site
- TELEPORT = “Dense Coding, Teleportation, No Cloning” Course web site

Types of quantum information: QIT. There is also a published paper “Types of Quantum Information” available on the course web site, which is much longer and more detailed. The relevant sections are I, II, and III.

Dense Coding: QCQI Sec. 2.3; TELEPORT; The original paper by Bennett and Wiesner, reference in QCQI p. 119, is also worth reading

No cloning: QCQI Sec. 12.1, and Proposition 12.18 in Sec. 12.6.3; TELEPORT

Teleportation: QCQI Secs. 1.3.7, 4.4; TELEPORT; Bennett et al., Phys. Rev. Lett. 70 (1993) 1895 (Original paper). If you are ambitious, take a look at R. B. Griffiths, Phys. Rev. A 66 (2002) 012311, also in <http://arxiv.org/archive/quant-ph/0203058>.

READING AHEAD:

- Cryptography: Stinson, *Cryptography: theory and practice*, Ch. 1; QCQI Sec. 12.6.1.
- Number-theoretic algorithms: CLRS Ch. 31; DPV Ch. 1; QCQI App. 4
- Complexity theory: CLRS Ch. 34; DPV Ch. 8; QCQI Ch. 3

EXERCISES (Not to be turned in)

1: From a previous hour exam.

**INSTRUCTIONS.** This examination consists of one question with 6 parts, each of which is worth between 15 and 20 points. If you cannot make progress on one part, take a look at a later part, for you may be able to do something with it. As on all examinations, *it is important to give a (brief!) indication of your reasoning.* Answer in a way which shows that you *understand* the subject, you really *do* know what you are doing.

There is a short list of formulas on the back side. If you think something else should be there, ask the instructor.

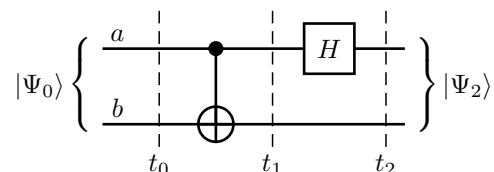
The circuit in the figure results in the state

$$|\Psi_2\rangle = \sqrt{1/3} |01\rangle + \sqrt{2/3} |10\rangle$$

at time  $t_2$ .

a) Find  $[0]_a \cdot |\Psi_2\rangle$  and  $[1]_a \cdot |\Psi_2\rangle$  (where as usual we think of  $[0]_a$  as denoting  $[0]_a \otimes I_b$ ) and use these to calculate the probabilities that qubit  $a$  at time  $t_2$  is in the state  $|0\rangle$  or  $|1\rangle$ .

b) Find the reduced density operator  $\rho_a$  for qubit  $a$  at time  $t_2$ . What is its geometrical representation as a point inside the Bloch sphere? (I.e., where in the Bloch sphere is the point located?)



c) Use  $\rho_a$  to calculate the same probabilities you calculated in (a), indicating how you use  $\rho_a$  to obtain them.

d) Starting from  $|\Psi_2\rangle$  use the circuit in the figure to calculate  $|\Psi_1\rangle$  and  $|\Psi_0\rangle$  at the earlier times  $t_1$  and  $t_0$ .

e) Consider a family of histories for the times  $t_0 < t_1 < t_2$  of the form

$$[\Psi_0] \odot \{P, \tilde{P}\} \odot \{Q, \tilde{Q}\},$$

where  $\tilde{P} = I - P$  and  $\tilde{Q} = I - Q$ , thus four histories, all with the same initial state  $[\Psi_0]$ . Explain why this family is consistent if  $P = [\Psi_1]$ , whatever the choice of the projector  $Q$ . You do not have to (and should not) make any reference to the specific  $|\Psi_1\rangle$  you computed in (d).

f) Show that the family in (e) is consistent if  $P = [ + ]_a$  and  $Q = [ 0 ]_a$ . ( $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ .)

2. a) Find the normalized kets, expressed as linear combinations of  $|0\rangle$  and  $|1\rangle$ , which in the Bloch sphere (spin half) picture correspond to the directions

$$w_1 = (\hat{x} + \hat{y} + \hat{z})/\sqrt{3}, \quad w_2 = (\hat{x} + \hat{y} - \hat{z})/\sqrt{3}, \quad w_3 = (-\hat{x} - \hat{y} - \hat{z})/\sqrt{3}.$$

That is, find  $|w^+\rangle$  for each direction, with numerical values for the coefficients of  $|0\rangle$  and  $|1\rangle$  (three decimal places is sufficient). Check that  $\langle w_3 | w_1 \rangle$  has the value you expect.

b) Find the directions,  $\theta$  and  $\phi$ , on the Bloch sphere corresponding to the following kets:

$$\begin{aligned} |\psi_1\rangle &= -|0\rangle + 1/\sqrt{3}|1\rangle \\ |\psi_2\rangle &= -i|0\rangle + 1/\sqrt{3}|1\rangle \\ |\psi_3\rangle &= (1+i)|0\rangle + (1-i)|1\rangle \end{aligned}$$

3. a) Let  $T$  be the time transformation operator from  $t_0$  to  $t_1$ , and suppose that at  $t_0$  a qubit is in the state  $|0\rangle$ . Find a unitary transformation  $T$  such that at  $t_1$  the probabilities of being in the state  $|0\rangle$  and  $|1\rangle$  are

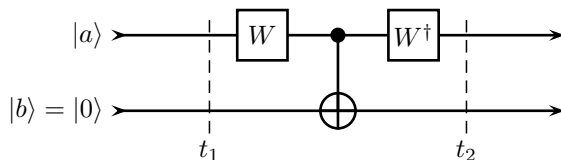
$$\Pr(0_1|0_0) = 2/3; \quad \Pr(1_1|0_0) = 1/3.$$

Here the subscripts label the time. (The answer is not unique, but a simple choice will simplify the next question.)

b) Describe  $T$  as a rotation on the Bloch sphere: some angle in a particular sense about some axis.

c) For the same unitary compute  $\Pr(0_1|1_0)$  and  $\Pr(1_1|1_0)$ , i.e., the probabilities if the initial state is  $|1\rangle$ . If you have done the calculation correctly there will be a surprising connection between these numbers and those in (a). Show that this connection is not accidental, but is implied by the unitarity of  $T$ .

4. a) Show that the following circuit can be used to measure qubit  $a$  in the orthonormal basis  $|w^+\rangle, |w^-\rangle$  provided  $W$  is chosen appropriately, assuming a pointer basis  $|0\rangle, |1\rangle$  for qubit  $b$ . (You may find it convenient to express  $W$  in terms of dyads.)



Assume an initial state

$$|\Psi_0\rangle = (\alpha|w^+\rangle + \beta|w^-\rangle) \otimes |0\rangle,$$

at  $t_0$ . Indicate a consistent family of histories (and show that the family is consistent) so that the measurement outcome,  $[0]_b$  or  $[1]_b$ , at time  $t_2$  is appropriately correlated with properties of qubit  $a$  at time  $t_1$  just before the measurement.

b) What are the states of  $a$  at  $t_2$  conditioned on the  $b$  pointer states at this time?

c) Would your answers to (a) and (b) be different if the  $W^\dagger$  gate were removed (set equal to  $I$ )?