Chapter 23

Singlet State Correlations

23.1 Introduction

This and the following chapter can be thought of as a single unit devoted to discussing various issues raised by a famous paper published by Einstein, Podolsky, and Rosen in 1935, in which they claimed to show that quantum mechanics, as it was understood at that time, was an incomplete theory. In particular, they asserted that a quantum wave function cannot provide a complete description of a quantum system. What they were concerned with was the problem of assigning simultaneous values to non-commuting operators, a topic which has already been discussed to some extent in Ch. 22. Their strategy was to consider an entangled state (see the definition in Sec. 6.2) of two spatially separated systems, and they argued that by carrying out a measurement on one system it was possible to determine a property of the other.

A simple example of an entangled state of spatially separated systems involves the spin degrees of freedom of two spin-half particles that are in different regions of space. In 1951 Bohm pointed out that the claim of the Einstein, Podolsky, and Rosen paper, commonly referred to as EPR, could be formulated in a simple way in terms of a singlet state of two spins, as defined in (23.2) below. Much of the subsequent discussion of the EPR problem has followed Bohm's lead, and that is the approach adopted in this and the following chapter. In this chapter we shall discuss various histories for two spin-half particles initially in a singlet state, and pay particular attention to the statistical correlations between the two spins. The basic correlation function which enters many discussions of the EPR problem is evaluated in Sec. 23.2 using histories involving just two times. A number of families of histories involving three times are considered in Sec. 23.3, while Sec. 23.4 discusses what happens when a spin measurement is carried out on one particle, and Sec. 23.5 the case of measurements of both particles.

The results found in this chapter may seem a bit dull and repetitious, and the reader who finds them so should skip ahead to the next chapter where the EPR problem itself, in Bohm's formulation, is stated in Sec. 24.1 in the form of a paradox, and the paradox is explored using various results derived in the present chapter. An alternative way of looking at the paradox using counterfactuals is discussed in Sec. 24.2. The remainder of Ch. 24 deals with an alternative approach to the EPR problem in which one adds an additional mathematical structure, usually referred to as "hidden variables", to the standard quantum Hilbert space of wave functions. A simple example of hidden variables in the context of measurements on particles in a spin singlet state, due to Mermin, is the topic of Sec. 24.3. It disagrees with the predictions of quantum theory for the spin correlation function, and this disagreement is not a coincidence, for Bell has shown by means of an inequality that *any* hidden variables theory of this sort *must* disagree with the predictions of quantum theory. The derivation of this inequality is taken up in Sec. 24.4, which also contains some remarks on its significance for the (non)existence of mysterious nonlocal influences in the quantum world.

23.2 Spin Correlations

Imagine two spin-half particles a and b traveling away from each other in a region of zero magnetic field (so the spin direction of each particle will remain fixed), and described by a wave function

$$|\chi_t\rangle = |\psi_0\rangle \otimes |\omega_t\rangle, \tag{23.1}$$

where $|\omega_t\rangle$ is a wave packet $\omega(\boldsymbol{r}_a, \boldsymbol{r}_b, t)$ describing the positions of the two particles, while

$$|\psi_0\rangle = \big(|z_a^+\rangle|z_b^-\rangle - |z_a^-\rangle|z_b^+\rangle\big)/\sqrt{2} \tag{23.2}$$

is the singlet state of the spins of the two particles, the state with total angular momentum equal to zero. Hereafter we shall ignore $|\omega_t\rangle$, as it plays no essential role in the following arguments, and concentrate on the spin state $|\psi_0\rangle$.

Rather than using eigenstates of S_{az} and S_{bz}, $|\psi_0\rangle$ can be written equally well in terms of eigenstates of S_{aw} and S_{bw}, where w is some direction in space described by the polar angles ϑ and φ. The states $|w^+\rangle$ and $|w^-\rangle$ are given as linear combinations of $|z^+\rangle$ and $|z^-\rangle$ in (4.14), and using these expressions one can rewrite $|\psi_0\rangle$ in the form

$$|\psi_0\rangle = (|w_a^+\rangle|w_b^-\rangle - |w_a^-\rangle|w_b^+\rangle)/\sqrt{2}, \tag{23.3}$$

or as

$$\begin{aligned} |\psi_0\rangle = \sin(\vartheta/2)&\Big[e^{-i\varphi/2}|z_a^+\rangle|w_b^+\rangle + e^{i\varphi/2}|z_a^-\rangle|w_b^-\rangle\Big]/\sqrt{2} \\ + \cos(\vartheta/2)&\Big[e^{-i\varphi/2}|z_a^+\rangle|w_b^-\rangle - e^{i\varphi/2}|z_a^-\rangle|w_b^+\rangle\Big]/\sqrt{2}, \end{aligned} \tag{23.4}$$

where ϑ and φ are the polar angles for the direction w, with w the positive z axis when $\vartheta = 0$. The fact that $|\psi_0\rangle$ has the same functional form in (23.3) as in (23.2) reflects the fact that this state is spherically symmetrical, and thus does not single out any particular direction in space.

Consider the consistent family whose support is a set of four histories at the two times $t_0 < t_1$:

$$\psi_0 \odot \{z_a^+, z_a^-\}\{w_b^+, w_b^-\}, \tag{23.5}$$

where the product of the two curly brackets stands for the set of four projectors $z_a^+ w_b^+$, $z_a^+ w_b^-$, $z_a^- w_b^+$, and $z_a^- w_b^-$. The time development operator $T(t_1, t_0)$ is equal to I, since we are only considering the spins and not the spatial wave function $\omega(\boldsymbol{r}_a, \boldsymbol{r}_b, t)$. Thus one can calculate the probabilities of these histories, or of the events at t_1 given ψ_0 at t_0, by thinking of $|\psi_0\rangle$ in (23.4) as a pre-probability and using the absolute squares of the corresponding coefficients. The result is:

$$\begin{aligned} \Pr(z_a^+, w_b^+) = \Pr(z_a^-, w_b^-) = \tfrac{1}{2}\sin^2(\vartheta/2) = (1-\cos\vartheta)/4, \\ \Pr(z_a^+, w_b^-) = \Pr(z_a^-, w_b^+) = \tfrac{1}{2}\cos^2(\vartheta/2) = (1+\cos\vartheta)/4, \end{aligned} \tag{23.6}$$

where one could also write $\Pr(z_a^+ \wedge w_b^+)$ in place of $\Pr(z_a^+, w_b^+)$ for the probability of $S_{az} = +1/2$ and $S_{bw} = +1/2$. Using these probabilities one can evaluate the *correlation function*

$$\begin{gathered} C(z,w) = \langle (2S_{az})(2S_{bw})\rangle = 4\langle\psi_0|S_{az}S_{bw}|\psi_0\rangle = \\ \Pr(z_a^+, w_b^+) + \Pr(z_a^-, w_b^-) - \Pr(z_a^+, w_b^-) - \Pr(z_a^-, w_b^+) = -\cos\vartheta. \end{gathered} \tag{23.7}$$

Because $|\psi_0\rangle$ is spherically symmetrical, one can immediately generalize these results to the case of a family of histories in which the directions z and w in (23.5) are replaced by arbitrary directions w_a and w_b, which can conveniently be written in the form of unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Since the cosine of the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to the dot product $\boldsymbol{a}\cdot\boldsymbol{b}$, the generalization of (23.6) is

$$\begin{aligned} \Pr(\boldsymbol{a}^+, \boldsymbol{b}^+) = \Pr(\boldsymbol{a}^-, \boldsymbol{b}^-) = (1-\boldsymbol{a}\cdot\boldsymbol{b})/2, \\ \Pr(\boldsymbol{a}^+, \boldsymbol{b}^-) = \Pr(\boldsymbol{a}^-, \boldsymbol{b}^+) = (1+\boldsymbol{a}\cdot\boldsymbol{b})/2, \end{aligned} \tag{23.8}$$

while the correlation function (23.7) is given by

$$C(\boldsymbol{a},\boldsymbol{b}) = -\boldsymbol{a}\cdot\boldsymbol{b}. \tag{23.9}$$

As will be shown in Sec. 23.5, $C(\boldsymbol{a},\boldsymbol{b})$ is also the correlation function for the outcomes, expressed in a suitable way, of measurements of the spin components of particles a and b in the directions $\boldsymbol{a}$ and $\boldsymbol{b}$.

23.3 Histories for Three Times

Let us now consider various families of histories for the times $t_0 < t_1 < t_2$, assuming an initial state ψ_0 at t_0. One possibility is a unitary history with ψ_0 at all three times, but in addition there are various stochastic histories. As a first example, consider the consistent family whose support consists of the two histories

$$\psi_0 \odot \begin{cases} z_a^+ z_b^- \odot z_a^+ z_b^-, \\ z_a^- z_b^+ \odot z_a^- z_b^+. \end{cases} \tag{23.10}$$

Each history carries a weight of 1/2 and describes a situation in which $S_{bz} = -S_{az}$, with values which are independent of time for $t > t_0$. In particular, one has conditional probabilities

$$\Pr(z_{a1}^+ \,|\, z_{a2}^+) = \Pr(z_{b1}^- \,|\, z_{a2}^+) = \Pr(z_{b2}^- \,|\, z_{a2}^+) = 1, \tag{23.11}$$

$$\Pr(z_{a1}^- \,|\, z_{b1}^+) = \Pr(z_{a2}^- \,|\, z_{b1}^+) = \Pr(z_{b2}^+ \,|\, z_{b1}^+) = 1, \tag{23.12}$$

among others, where the time, t_1 or t_2, at which an event occurs is indicated by a subscript 1 or 2. Thus if $S_{az} = +1/2$ at t_2, then it had this same value at t_1, and one can be certain that S_{bz} has the value $-1/2$ at both t_1 and t_2.

Because of spherical symmetry, the same sort of family can be constructed with z replaced by an arbitrary direction w. In particular, with $w = x$, we have a family with support

$$\psi_0 \odot \begin{cases} x_a^+ x_b^- \odot x_a^+ x_b^-, \\ x_a^- x_b^+ \odot x_a^- x_b^+. \end{cases} \tag{23.13}$$

Again, each history has a weight of 1/2, and now it is the values of S_{ax} and S_{bx} which are of opposite sign and independent of time, and the results in (23.11) and (23.12) hold with z replaced by x. The two families (23.10) and (23.13) are obviously incompatible with each other because the projectors for one family do not commute with those of the other. There is no way in which they can be combined in a single description, and the corresponding conditional probabilities cannot be related to one another, since they are defined on separate sample spaces.

One can also consider a family in which a stochastic branching takes place between t_1 and t_2 instead of between t_0 and t_1; thus (23.10) can be replaced with

$$\psi_0 \odot \psi_0 \odot \{z_a^+ z_b^-, z_a^- z_b^+\}. \tag{23.14}$$

In this case the last equality in (23.11) remains valid, but the other conditional probabilities in (23.11) and (23.12) are undefined, because (23.14) does not contain projectors corresponding to values of S_{az} and S_{bz} at time t_1, and they cannot be added to this family, as they do not commute with ψ_0.

One need not limit oneself to families in which the same component of spin angular momentum is employed for both particles. The four histories

$$\psi_0 \odot \begin{cases} z_a^+ x_b^+ \odot z_a^+ x_b^+, \\ z_a^+ x_b^- \odot z_a^+ x_b^-, \\ z_a^- x_b^+ \odot z_a^- x_b^+, \\ z_a^- x_b^- \odot z_a^- x_b^- \end{cases} \tag{23.15}$$

form the support of a consistent family. Since they all have equal weight, one has conditional probabilities

$$\Pr(x_b^+ \mid z_a^+) = 1/2 = \Pr(x_b^- \mid z_a^+), \tag{23.16}$$

and others of a similar type which hold for events at both t_1 and t_2, which is why subscripts 1 and 2 have been omitted. In addition, the values of S_{az} and S_{bx} do not change with time:

$$\begin{aligned} \Pr(z_{a2}^+ \mid z_{a1}^+) &= 1 = \Pr(z_{a2}^- \mid z_{a1}^-), \\ \Pr(x_{b2}^+ \mid x_{b1}^+) &= 1 = \Pr(x_{b2}^- \mid x_{b1}^-). \end{aligned} \tag{23.17}$$

Yet another consistent family, with support

$$\psi_0 \odot \begin{cases} z_a^+ z_b^- \odot z_a^+ \{x_b^+, x_b^-\}, \\ z_a^- z_b^+ \odot z_a^- \{x_b^+, x_b^-\}, \end{cases} \tag{23.18}$$

where $z_a^+\{x_b^+, x_b^-\}$ denotes the pair of projectors $z_a^+ x_b^+$ and $z_a^+ x_b^-$, combines features of (23.10) and (23.15): values of S_{az} are part of the description at both t_1 and t_2, but in the case of particle b, two separate components, S_{bz} and S_{bx} are employed at t_1 and t_2. It is important to notice that this change is *not* brought about by any dynamical effect; instead, it is simply a consequence of using (23.18) rather than (23.10) or (23.15) as the framework for constructing the stochastic description. In particular, one can have a history in which $S_{bz} = +1/2$ at t_1 and $S_{bx} = -1/2$ at t_2. This does not mean that some torque is present which rotates the direction of the spin from the $+z$ to the $-x$ direction, for there is nothing which could produce such a torque. See the discussion following (9.33) in Sec. 9.3.

The families of histories considered thus far all satisfy the consistency conditions, as is clear from the fact that the final projectors are mutually orthogonal. Given that three times are involved, inconsistent families are also possible. Here is one which will be discussed later from the point of view of measurements. It contains the sixteen histories which can be represented in the compact form

$$\psi_0 \odot \{x_a^+, x_a^-\}\{z_b^+, z_b^-\} \odot \{z_a^+, z_a^-\}\{x_b^+, x_b^-\}, \tag{23.19}$$

where the product of curly brackets at each of the two times stands for a collection of four projectors, as in (23.5). Each history makes use of one of the four projectors at each of the two times; for example,

$$\psi_o \odot x_a^+ z_b^- \odot z_a^- x_b^- \tag{23.20}$$

is one of the sixteen histories. Each of these histories has a finite weight, and the chain kets of the four histories ending in $z_a^- x_b^-$, to take an example, are all proportional to $|z_a^-\rangle|x_b^-\rangle$, so cannot be orthogonal to each other.

23.4 Measurements of One Spin

Suppose that the z component S_{az} of the spin of particle a is measured using a Stern-Gerlach apparatus as discussed in Ch. 17. The initial state of the apparatus is $|Z_a^\circ\rangle$, and its interaction with the particle during the time interval from t_1 to t_2 gives rise to a unitary time evolution

$$|z_a^+\rangle|Z_a^\circ\rangle \mapsto |Z_a^+\rangle, \quad |z_a^-\rangle|Z_a^\circ\rangle \mapsto |Z_a^-\rangle, \tag{23.21}$$

where $|Z_a^+\rangle$ and $|Z_a^-\rangle$ are apparatus states ("pointer positions") indicating the two possible outcomes of the measurement. Note that the spin states no longer appear on the right side; we are assuming that at t_2 the spin-half particle has become part of the measuring apparatus. (Thus (23.21) represents a destructive measurement in the terminology of Sec. 17.1. One could also consider nondestructive measurements in which the value of S_{az} is the same after the measurement as it is before, by using (18.17) in place of (23.21), but these will not be needed for the following discussion.) The b particle has no effect on the apparatus, and vice versa. That is, one can place an arbitrary spin state $|w_b^+\rangle$ for the b particle on both sides of the arrows in (23.21).

Consider the consistent family with support

$$\Psi_0^z \odot \begin{cases} z_a^+ z_b^- \odot Z_a^+ z_b^-, \\ z_a^- z_b^+ \odot Z_a^- z_b^+, \end{cases} \tag{23.22}$$

where the initial state is $|\Psi_0^z\rangle = |\psi_0\rangle|Z_a^\circ\rangle$. The conditional probabilities

$$\begin{aligned}
\Pr(z_{a1}^+ \,|\, Z_{a2}^+) &= 1 = \Pr(z_{a1}^- \,|\, Z_{a2}^-), && (23.23)\\
\Pr(z_{b1}^- \,|\, Z_{a2}^+) &= 1 = \Pr(z_{b1}^+ \,|\, Z_{a2}^-), && (23.24)\\
\Pr(z_{b2}^- \,|\, Z_{a2}^+) &= 1 = \Pr(z_{b2}^+ \,|\, Z_{a2}^-), && (23.25)\\
\Pr(z_{b2}^+ \,|\, z_{b1}^+) &= 1 = \Pr(z_{b2}^- \,|\, z_{b1}^-) && (23.26)
\end{aligned}$$

are an obvious consequence of (23.22). The first pair, (23.23), tell us that the measurement is, indeed, a measurement: the outcomes $Z^\pm$ actually reveal values of S_{az} before the measurement took place. Those in (23.24) and (23.25) tell us that the measurement is also an *indirect* measurement of S_{bz} for particle b, even though this particle never interacts with the apparatus that measures S_{az}, since the measurement outcomes Z_a^+ and Z_a^- are correlated with the properties z_b^- and z_b^+.

There is nothing very surprising about carrying out an indirect measurement of the property of a distant object in this way, and the ability to do so does not indicate any sort of mysterious long-range or nonlocal influence. Consider the following analogy. Two slips of paper, one red and one green, are placed in separate opaque envelopes. One envelope is mailed to a scientist in Atlanta and the other to a scientist in Boston. When the scientist in Atlanta opens the envelope and looks at the slip of paper, he can immediately infer the color of the slip in the envelope in Boston, and for this reason he has, in effect, carried out an indirect measurement. Furthermore, this measurement indicates the color of the slip of paper in Boston not only at the time the measurement is carried out, but also at earlier and later times, assuming the slip in Boston does not undergo some process which changes its color. In the same way, the outcome, Z_a^+ or Z_a^-, for the measurement of S_{az} allows one to infer the value of S_{bz} both at t_1 and at t_2, and at later times as well if one extends the histories in (23.22) in an appropriate manner. In order for this inference to be correct, it is necessary that particle b not interact with anything, such as a measuring device or magnetic field, which could perturb its spin.

The conditional probabilities in (23.26) tell us that S_{bz} is the same at t_2 as at t_1, consistent with our assumption that particle b has not interacted with anything during this time interval. Note, in particular, that carrying out a measurement on S_{az} has no influence on S_{bz}, which is just what one would expect, since particle b is isolated from particle a, and from the measuring apparatus, at all times later than t_0.

A similar discussion applies to a measurement carried out on some other component of the spin of particle a. To measure S_{ax}, what one needs is an apparatus initially in the state $|X_a^\circ\rangle$, which during the time interval from t_1 to t_2 interacts with particle a in such a way as to give rise to the unitary time transformation

$$|x_a^+\rangle|X_a^\circ\rangle \mapsto |x_a^+\rangle|X_a^+\rangle, \quad |x_a^-\rangle|X_a^\circ\rangle \mapsto |x_a^-\rangle|X_a^-\rangle. \tag{23.27}$$

The counterpart of (23.22) is the consistent family with support

$$\Psi_0^x \odot \begin{cases} x_a^+ x_b^- \odot X_a^+ x_b^-, \\ x_a^- x_b^+ \odot X_a^- x_b^+, \end{cases} \tag{23.28}$$

where the initial state is now $|\Psi_0^x\rangle = |\psi_0\rangle|X_a^\circ\rangle$. Using this family, one can calculate probabilities analogous to those in (23.23) to (23.26), with z and Z replaced by x and X. Thus in this framework a measurement of S_{ax} is an indirect measurement of S_{bx}, and one can show that the measurement has no effect upon S_{bx}.

Comparing (23.22) with (23.10), or (23.28) with (23.13) shows that the families which describe measurement results are close parallels of those describing the system of two spins in the absence of any measurements. To include the measurement, one simply introduces an appropriate initial state at t_0, and replaces one of the lower case letters at t_2 with the corresponding capital to indicate a measurement outcome. This should come as no surprise: apparatus designed to measure some property will, if it is working properly, measure that property. Once one knows how to describe a quantum system in terms of its microscopic properties, the addition of a measurement apparatus of an appropriate type will simply confirm the correctness of the microscopic description.

Replacing lower case with capital letters can also be used to construct measurement counterparts of other consistent families in Sec. 23.3. The counterpart of (23.14) when S_{az} is measured is the family with support

$$\Psi_0^z \odot \Psi_0^z \odot \{Z_a^+ z_b^-, Z_a^- z_b^+\}. \tag{23.29}$$

Using this family one can deduce the conditional probabilities in (23.25) referring to the values of S_{bz} at t_2, and thus the measurement of S_{az}, viewed within this framework, is again an indirect measurement of S_{bz} at t_2. However, the results in (23.23), (23.24), and (23.26) are not valid for the family (23.29), because values of S_{az} and S_{bz} cannot be defined at t_1: the corresponding projectors do not commute with the ψ_0 part of Ψ_0^z.

One reason for introducing (23.29) is that it is the family which comes closest to representing the idea that a measurement is associated with a collapse of the wave function of the measured system. In the case at hand, the measured system can be thought of as the spin state of the two particles, but since particle a is no longer relevant to the discussion at t_2, collapse should be thought of as resulting in a state $|z_b^-\rangle$ or $|z_b^+\rangle$ for particle b, depending upon whether the measurement outcome is Z_a^+ or Z_a^-. (In the case of a nondestructive measurement on particle a the states resulting from the collapse would be $|z_a^+\rangle|z_b^-\rangle$ and $|z_a^-\rangle|z_b^+\rangle$.) As pointed out in Sec. 18.2, wave function collapse is basically a mathematical procedure for computing certain types of conditional probabilities. Regarding it as some sort of physical process gives rise to a misleading picture of instantaneous influences which can travel faster than the speed of light. The remarks in Sec. 18.2 with reference to the beam splitter in Fig. 18.1 apply equally well to spatially separated systems of spin-half particles, or of photons, etc.

One way to see that the measurement of S_{az} is not a process which somehow brings S_{bz} into existence at t_2 is to note that the change between t_1 and the final time t_2 in (23.29) is similar to the change which occurs in the family (23.14), where there is no measurement. Another way to see this is to consider the family whose support consists of the four histories

$$\Psi_0^z \odot \Psi_0^z \odot \{Z_a^+, Z_a^-\}\{x_b^+, x_b^-\} \tag{23.30}$$

in the compact notation used earlier in (23.5). This resembles (23.29), except that the components of S_{bx} rather than S_{bz} appear at t_2. Were the measurement having some physical effect on particle b, it would be just as sensible to suppose that it produces random values of S_{bx}, as that it results in a value of S_{bz} correlated with the outcome of the measurement!

It was noted earlier that (23.26) implies that measuring S_{az} has no effect upon S_{bz}. Nor does such a measurement influence any other component of the spin of particle b, as can be seen by constructing an appropriate consistent family in which this component enters the description at both t_1 and t_2. Thus in the case of S_{bx} one can use the measurement counterpart of (23.15), a family with support

$$\Psi_0^z \odot \begin{cases} z_a^+ x_b^+ \odot Z_a^+ x_b^+, \\ z_a^+ x_b^- \odot Z_a^+ x_b^-, \\ z_a^- x_b^+ \odot Z_a^- x_b^+, \\ z_a^- x_b^- \odot Z_a^- x_b^-. \end{cases} \tag{23.31}$$

It is then evident by inspection that S_{bx} is the same at t_1 and t_2. Using this family one obtains the conditional probabilities

$$\begin{aligned} \Pr(x_b^+ \mid Z_{a2}^+) = 1/2 = \Pr(x_b^- \mid Z_{a2}^+), \\ \Pr(x_b^+ \mid Z_{a2}^-) = 1/2 = \Pr(x_b^- \mid Z_{a2}^-), \end{aligned} \tag{23.32}$$

where the subscript indicating the time has been omitted from $x_b^\pm$, since these results apply equally at t_1 and t_2. Of course (23.32) is nothing but the measurement counterpart of (23.16). It tells one that a measurement of S_{az} can in no way be regarded as an indirect measurement of S_{bx}. Similar results are obtained if the projectors corresponding to S_{bx} in (23.31) are replaced by those corresponding to S_{bw} for some other direction w, except that the conditional probabilities for w_b^+ and w_b^- in the expression corresponding to (23.32) will depend upon w. If w is close to z, a measurement of S_{az} is an approximate indirect measurement of S_{bw} in the sense that $S_{bw} = -S_{az}$ for most experimental runs, with occasional errors.

The family

$$\Psi_0^z \odot \begin{cases} z_a^+ z_b^- \odot Z_a^+ \{x_b^+, x_b^-\}, \\ z_a^- z_b^+ \odot Z_a^- \{x_b^+, x_b^-\} \end{cases} \tag{23.33}$$

is the counterpart of (23.18) when S_{az} is measured. Here the events involving the spin of particle b are different at t_2 from what they are at t_1. However, just as in the case of (23.18), for which no measurement occurs, one should not think of this change as a physical consequence of the measurement. See the discussion following (23.18).

23.5 Measurements of Two Spins

Thus far we have only considered measurements on particle a. One can also imagine carrying out measurements on the spins of both particles. All that is needed is a second measuring device of a type appropriate for whatever component of the spin of particle b is of interest. If, for example, this is S_{bx}, then the unitary time transformation from t_1 to t_2 will be the same as (23.27) except for replacing the subscript a with b. In what follows it will be convenient to assume that measurements are carried out on both particles at the same time. However, this is not essential; analogous results are obtained if measurements are carried out at different times. The properties of a particle will, in general, be different before and after it is measured, but the time at which a measurement is carried out on the other particle is completely irrelevant.

For the combined system of two particles and two measuring devices a typical unitary transformation from t_1 to t_2 takes the form:

$$|z_a^+\rangle|x_b^-\rangle|Z_a^\circ\rangle|X_b^\circ\rangle \mapsto |Z_a^+\rangle|X_b^-\rangle. \tag{23.34}$$

Once again, one can generate consistent families for measurements by starting off with any of the consistent families in Sec. 23.3, replacing ψ_0 with an appropriate initial state which includes each apparatus in its ready state, and then replacing lower case letters at the final time t_2 with corresponding capitals. For example

$$\Psi_0^{zz} \odot \begin{cases} z_a^+ z_b^- \odot Z_a^+ Z_b^-, \\ z_a^- z_b^+ \odot Z_a^- Z_b^+, \end{cases} \tag{23.35}$$

with $|\Psi_0^{zz}\rangle = |\psi_0\rangle|Z_a^\circ\rangle|Z_b^\circ\rangle$, is the counterpart of (23.10), and it shows that the outcomes of measurements of S_{az} and S_{bz} will be perfectly anticorrelated:

$$\Pr(Z_b^- \mid Z_a^+) = 1 = \Pr(Z_b^+ \mid Z_a^-). \tag{23.36}$$

Not only does one obtain consistent families by this process of "capitalizing" those in Sec. 23.3, the *weights* for histories involving measurements are also precisely the same as their counterparts that involve only particle properties. This means that the correlation function $C(\boldsymbol{a}, \boldsymbol{b})$ introduced in Sec. 23.1 can be applied to measurement outcomes as well as to microscopic properties. To do this, let $\alpha(w)$ be $+1$ if the apparatus designed to measure S_{aw} is in the state $|W_a^+\rangle$ at t_2, and -1 if it is in the state $|W_a^-\rangle$, and define $\beta(w)$ in the same way for measurements on particle b. Then we can write

$$C(\boldsymbol{a}, \boldsymbol{b}) = \langle \alpha(w_a)\beta(w_b)\rangle = -\boldsymbol{a} \cdot \boldsymbol{b} \tag{23.37}$$

as the average over a large number of experimental runs of the product $\alpha(w_a)\beta(w_b)$ when w_a is $\boldsymbol{a}$ and w_b is $\boldsymbol{b}$.

The physical significance of C in (23.37) is, of course, different from that in (23.9). The former refers to measurement outcomes and the latter to properties of the two particles. However, they are identical functions of $\boldsymbol{a}$ and $\boldsymbol{b}$, and given that the measurements accurately reflect previous values of the corresponding spin components, no confusion will arise from using the same symbol in both cases. One could also, to be sure, define the same sort of correlation for a case in which a spin component is measured for only one particle, using the product of the outcome of that measurement, understood as ± 1, with twice the value (in units of $\hbar$) of the appropriate spin component for the other particle; for example

$$C(w, w') = \langle \alpha(w) 2S_{bw'}\rangle. \tag{23.38}$$

As noted in Sec. 23.4, the outcome of a measurement of the z component of the spin of particle a can be used to infer the value of S_{az} before the measurement, and the value of S_{bz} for particle b as long as that particle remains isolated. The roles of particle a and b can be interchanged: a measurement of S_{bz} for particle b allows one to infer the value of S_{az}. And because of the spherical symmetry of ψ_0, the same results hold if z is replaced by any other direction w. How are these results modified, or extended, if the spins of *both* particles are measured? If the same component of spin is measured for particle b as for particle a, the results are just what one would expect. Suppose it is the z component. Then (23.35) shows that one can infer both z_a^+ and z_b^- on the basis of the outcome Z_a^+, or of the outcome Z_b^-, a result which is not surprising since one outcome implies the other, (23.36).

Things become more complicated if the a and b measurements involve different components, and in this case it is necessary to pay careful attention to the framework one is using for inferring microscopic properties from the outcomes of the measurements. To illustrate this, let us suppose that S_{az} is measured for particle a and S_{bx} for particle b. One consistent family that can be used for analyzing this situation is the counterpart of (23.15):

$$\Psi_0^{zx} \odot \begin{cases} z_a^+ x_b^+ \odot Z_a^+ X_b^+, \\ z_a^+ x_b^- \odot Z_a^+ X_b^-, \\ z_a^- x_b^+ \odot Z_a^- X_b^+, \\ z_a^- x_b^- \odot Z_a^- X_b^-. \end{cases} \tag{23.39}$$

Here the initial state is $|\Psi_0^{zx}\rangle = |\psi_0\rangle|Z_a^\circ\rangle|X_b^\circ\rangle$. Using this family allows one to infer from the outcome of each measurement something about the spin of the same particle at an earlier time, but nothing about the spin of the other particle. Thus one has

$$\Pr(z_{a1}^+ \,|\, Z_{a2}^+) = 1 = \Pr(z_{a1}^- \,|\, Z_{a2}^-), \tag{23.40}$$

$$\Pr(x_{b1}^+ \,|\, X_{b2}^+) = 1 = \Pr(x_{b1}^- \,|\, X_{b2}^-), \tag{23.41}$$

but there is no counterpart of (23.24) relating S_{bz} to $Z_a^\pm$, nor a way to relate S_{ax} to $X_b^\pm$, because the relevant projectors, such as z_b^+, are not present in (23.39) at t_1, nor can they be added, since they do not commute with the projectors which are already there.

On the other hand, the family with support

$$\Psi_0^{zx} \odot \begin{cases} z_a^+ z_b^- \odot Z_a^+ \{X_b^+, X_b^-\}, \\ z_a^- z_b^+ \odot Z_a^- \{X_b^+, X_b^-\}, \end{cases} \tag{23.42}$$

which is the counterpart of (23.18) and (23.33), can be used to infer values of S_{bz} from the outcomes $Z_a^\pm$. By using it, one obtains the conditional probabilities

$$\Pr(z_{b1}^- \,|\, Z_{a2}^+) = 1 = \Pr(z_{b1}^+ \,|\, Z_{a2}^-) \tag{23.43}$$

in addition to (23.40). However, if one uses (23.42) the outcome of the b measurement tells one nothing about S_{bx} at t_1. It is worth noting that a refinement of (23.42) in which additional events are added at a time $t_{1.5}$, so that the histories

$$\Psi_0^{zx} \odot \begin{cases} z_a^+ z_b^- \odot \begin{cases} z_a^+ x_b^+ \odot Z_a^+ X_b^+, \\ z_a^+ x_b^- \odot Z_a^+ X_b^-, \end{cases} \\ z_a^- z_b^+ \odot \begin{cases} z_a^- x_b^+ \odot Z_a^- X_b^+, \\ z_a^- x_b^- \odot Z_a^- X_b^- \end{cases} \end{cases} \tag{23.44}$$

are defined at $t_0 < t_1 < t_{1.5} < t_2$, is the support of a consistent family in which one can infer from X_b^+ or X_b^- at t_2 the value of S_{bx} at $t_{1.5}$, but not at an earlier time. As this is a refinement of (23.42), both (23.40) and (23.43) remain valid.

The consistent family with support

$$\Psi_0^{zx} \odot \begin{cases} x_a^+ x_b^- \odot \{Z_a^+, Z_a^-\} X_b^-, \\ x_a^- x_b^+ \odot \{Z_a^+, Z_a^-\} X_b^+ \end{cases} \tag{23.45}$$

is the counterpart of (23.42) with x rather than z components at t_1. One can use it to infer the x component of the spin of either particle at t_1 from the outcome of the S_{bx} measurement:

$$\Pr(x_{b1}^+ \,|\, X_{b2}^+) = 1 = \Pr(x_{b1}^- \,|\, X_{b2}^-), \tag{23.46}$$

$$\Pr(x_{a1}^- \,|\, X_{b2}^+) = 1 = \Pr(x_{a1}^+ \,|\, X_{b2}^-). \tag{23.47}$$

Given the conditional probabilities in (23.43) and (23.47), and no indication of the consistent families from which they were obtained, one might be tempted to combine them and draw the conclusion that for a run in which the measurement outcomes are, say, Z_a^+ and X_b^+ at t_2, both S_{ax} and S_{bz} had the value $-1/2$ at t_1:

$$\Pr(x_{a1}^- \wedge z_{b1}^- \,|\, Z_{a2}^+ \wedge X_{b2}^+) = 1. \tag{23.48}$$

This, however, is not correct. To begin with, the frameworks (23.42) and (23.45) are mutually incompatible because of the projectors at t_1, so they cannot be used to derive (23.48) by combining (23.43) with (23.47). Next, if one tries to construct a single consistent family in which it might be possible to derive (23.48), one runs into the following difficulty. A description which ascribes values to both S_{ax} and S_{bz} at t_1 requires a decomposition of the identity which includes the four projectors $x_a^+ z_b^+$, $x_a^+ z_b^-$, $x_a^- z_b^+$, and $x_a^- z_b^-$. This by itself is not a problem, but when combined with the four measurement outcomes, the result is the *inconsistent* family

$$\Psi_0^{zx} \odot \{x_a^+, x_a^-\}\{z_b^+, z_b^-\} \odot \{Z_a^+, Z_a^-\}\{X_b^+, X_b^-\} \tag{23.49}$$

obtained by replacing ψ_0 with Ψ_0^{zx} and capitalizing x and z at t_2 in (23.19). The same arguments used to show that (23.19) is inconsistent apply equally to (23.49); adding measurements does not improve things. Consequently, because it cannot be obtained using a consistent family, (23.48) is not a valid result.