Chapter 12

Examples of Consistent Families

12.1 Toy Beam Splitter

Beam splitters are employed in optics, in devices such as the Michelson and Mach-Zehnder interferometers, to split an incoming beam of light into two separate beams propagating perpendicular to each other. The analogous situation in a neutron interferometer is achieved using a single crystal of silicon as a beam splitter. The toy beam splitter in Fig. 12.1 can be thought of as a model of either an optical or a neutron beam splitter. It has two entrance channels (or ports) a and b, and two exit channels c and d. The sites are labeled by a pair mz, where m is an integer, and z is one of the four letters a, b, c, or d, indicating the channel in which the site is located.

The unitary time development operator is $T = S_b$, where the action of the operator S_b is given by

$$S_b|mz\rangle = |(m+1)z\rangle, \tag{12.1}$$

with the exceptions:

$$\begin{aligned} S_b|0a\rangle &= \bigl(+|1c\rangle + |1d\rangle\bigr)/\sqrt{2}, \\ S_b|0b\rangle &= \bigl(-|1c\rangle + |1d\rangle\bigr)/\sqrt{2}. \end{aligned} \tag{12.2}$$

The physical significance of the states $|0a\rangle$, $|1c\rangle$, etc. is not altered if they are multiplied by arbitrary phase factors, see Sec. 2.2, and this means that (12.2) is not the only possible way of representing the action of the beam splitter. One could equally well replace the states on the right side with

$$\bigl(i|1c\rangle + |1d\rangle\bigr)/\sqrt{2}, \quad \bigl(|1c\rangle + i|1d\rangle\bigr)/\sqrt{2}, \tag{12.3}$$

or make other choices for the phases. There are two other exceptions to (12.1) that are needed to supply the model with periodic boundary conditions which connect the c channel back into the a channel and the d channel back into the b channel (or c into b and d into a if one prefers). It is not necessary to write down a formula, since we shall only be interested in short time intervals during which the particle will not pass across the periodic boundaries and come back to the beam splitter. That S_b is unitary follows from the fact that it maps an orthonormal basis of the Hilbert space, namely the collection of all kets of the form $|mz\rangle$, onto another orthonormal basis of the same space; see Sec. 7.2.

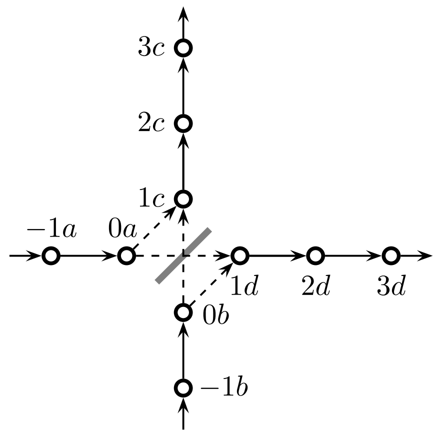

Figure 12.1: Toy beam splitter.

Suppose that at $t = 0$ the particle starts off in the state

$$|\psi_0\rangle = |0a\rangle, \tag{12.4}$$

that is, it is in the a channel and about to enter the beam splitter. Unitary time development up to a time $t > 0$ results in

$$|\psi_t\rangle = S_b^t|\psi_0\rangle = \big(|tc\rangle + |td\rangle\big)/\sqrt{2} = |t\bar{a}\rangle, \tag{12.5}$$

where

$$|m\bar{a}\rangle := \big(|mc\rangle + |md\rangle\big)/\sqrt{2}, \quad |m\bar{b}\rangle := \big(-|mc\rangle + |md\rangle\big)/\sqrt{2} \tag{12.6}$$

are the states resulting from unitary time evolution when the particle starts off in $|0a\rangle$ or $|0b\rangle$, respectively.

Let us consider histories involving just two times, with an initial state $|\psi_0\rangle = |0a\rangle$ at $t = 0$, and a basis at some time $t > 0$ consisting of the states $\{|mz\rangle\}$, $z = a$, b, c, or d, corresponding to a decomposition of the identity

$$I = \sum_{m,z} [mz]. \tag{12.7}$$

By treating $|\psi_t\rangle$ as a pre-probability, see Sec. 9.4, one finds that

$$\Pr([mc]_t) = (1/2)\delta_{tm} = \Pr([md]_t), \tag{12.8}$$

while all other probabilities vanish; i.e., at time t the particle will either be in the c output channel at the site tc, or in the d channel at td. Here $[mc]$ is a projector onto the ray which contains $|mc\rangle$, and the subscript indicates the time at which the event occurs.

If, on the other hand, one employs a unitary history, Sec. 8.7, in which at time t the particle is in the state $|t\bar{a}\rangle$, one cannot say that it is in either the c or the d channel. The situation is analogous to the case of a spin-half particle with an initial state $|z^+\rangle$ and trivial dynamics, discussed in Sec. 9.3. In a unitary history with $S_z = +1/2$ at a later time it is not meaningful to ascribe a value to S_x, whereas by using a sample space in which S_x at the later time makes sense, one concludes that $S_x = +1/2$ or $S_x = -1/2$, each with probability 1/2.

The toy beam splitter is a bit more complicated than a spin-half particle, because when we say that "the particle is in the c channel", we are not committed to saying that it is at a *particular site* in the c channel. Instead, being in the c channel, or being in the d channel is represented by means of projectors

$$C = \sum_m |mc\rangle\langle mc| = \sum_m [mc], \quad D = \sum_m [md]. \tag{12.9}$$

Neither of these projectors commutes with a projector $[m\bar{a}]$ corresponding to the state $|m\bar{a}\rangle$ defined in (12.6), so if we use a unitary history, we cannot say that the particle is in channel c or channel d. Note that whenever it *is* sensible to speak of a particle being in channel c or channel d, it cannot possibly be in both channels, since

$$CD = 0; \tag{12.10}$$

i.e., these properties are mutually exclusive. A quantum particle can lack a definite location, as in the state $|m\bar{a}\rangle$, but, as already pointed out in Sec. 4.5, it cannot be in two places at the same time.

The fact that the particle is at the site tc with probability 1/2 and at the site td with probability 1/2 at a time $t > 0$, (12.8), might suggest that with probability 1/2 the particle is moving out the c channel through a succession of sites $1c$, $2c$, $3c$, and so forth, and with probability 1/2 out the d channel through $1d$, $2d$, etc. But this is *not* something one can infer by considering histories defined at only two times, for it would be equally consistent to suppose that the particle hops from $2c$ to $3d$ during the time step from $t = 2$ to $t = 3$, and from $2d$ to $3c$ if it happens to be in the d channel at $t = 2$. In order to rule out unphysical possibilities of this sort we need to consider histories involving more than just two times.

Consider a family of histories based upon the initial state $[0a]$, and at each time $t > 0$ the decomposition of the identity (12.7), so that the particle has a definite location. The histories are then of the form, for a set of times $t = 0, 1, 2, \dots f$,

$$Y = [0a] \odot [mz] \odot [m'z'] \odot \cdots [m''z''], \tag{12.11}$$

with a chain operator of the form $K(Y) = |\phi\rangle\langle 0a|$, Sec. 11.6, where the chain ket is

$$|\phi\rangle = |m''z''\rangle \cdots \langle m'z'|S_b|mz\rangle\langle mz|S_b|0a\rangle. \tag{12.12}$$

From (12.2) it is obvious that the term $\langle mz|S_b|0a\rangle$ is zero unless $m = 1$ and $z = c$ or d, and given $m = 1$, it follows from (12.1) that $\langle m'z'|S_b|mz\rangle$ vanishes unless $m' = 2$ and $z' = z$. By continuing this argument one sees that $|\phi\rangle$, and therefore $K(Y)$, will vanish for all but two histories, which in the case $f = 4$ are

$$\begin{aligned} Y^c &= [0a] \odot [1c] \odot [2c] \odot [3c] \odot [4c], \\ Y^d &= [0a] \odot [1d] \odot [2d] \odot [3d] \odot [4d]. \end{aligned} \tag{12.13}$$

The fact that the final projectors $[4c]$ and $[4d]$ in (12.13) are orthogonal to each other means that the chain operators $K(Y^c)$ and $K(Y^d)$ are orthogonal, in accordance with a general principle noted in Sec. 11.3. Since the chain operators of all the other histories are zero, it follows that Y^c and Y^d form the support, as defined in Sec. 11.2, of a consistent family. It is straightforward to show, either by means of chain kets as discussed in Sec. 11.6 or by a direct use of $W(Y) = \langle K(Y), K(Y)\rangle$, that

$$W(Y^c) = 1/2 = W(Y^d), \tag{12.14}$$

and hence, assuming an initial state of $[0a]$ with probability 1, the two histories Y^c and Y^d each have probability 1/2, while all other histories in this family have probability zero.

The fact that the only histories with finite probability are Y^c and Y^d means that if the particle arrives at the site $1c$ at $t = 1$, it continues to move out along the c channel, and does not hop to the d channel, and if the particle is at $1d$ at time $t = 1$, it moves out along the d channel. Thus by using multiple-time histories one can eliminate the possibility that the particle hops back and forth between channels c and d, something which cannot be excluded by considering only two-time histories, as noted earlier. A formal argument confirming what is rather obvious from looking at (12.13) can be constructed by calculating the probability

$$\Pr(D_t \,|\, [1c]_1) = \Pr(D_t \wedge [1c]_1)/\Pr([1c]_1) \tag{12.15}$$

that the particle will be in the d channel at some time $t > 0$, given that it was at the site $[1c]$ at $t = 1$. Here D_t is a projector on the history space for the particle to be in channel d at time t. For example, for $t = 2$,

$$D_2 = I \odot I \odot D \odot I \odot I, \tag{12.16}$$

and thus

$$D_2 \wedge [1c]_1 = I \odot [1c] \odot D \odot I \odot I. \tag{12.17}$$

This projector gives zero when applied to either Y^c or Y^d, the only two histories with positive probability, and therefore the numerator on the right side of (12.15) is zero. Thus if the particle is at $1c$ at $t = 1$, it will not be in the d channel at $t = 2$. The same argument works equally well for other values of t, and analogous results are obtained if the particle is initially in the d channel. Thus one has

$$\begin{aligned} \Pr(D_t \,|\, [1c]_1) = 0 = \Pr(C_t \,|\, [1d]_1), \\ \Pr(C_t \,|\, [1c]_1) = 1 = \Pr(D_t \,|\, [1d]_1) \end{aligned} \tag{12.18}$$

for any $t \geq 1$, where C_t is defined in the same manner as D_t, with C in place of D.

(Since we are considering a family which is based on the initial state $[0a]$, the preceding discussion runs into the technical difficulty that C_t and D_t do not belong to the corresponding Boolean algebra of histories when the latter is constructed in the manner indicated in Sec. 8.5. One can get around this problem by replacing C_t and D_t with the operators $C_t \wedge [a0]_0$ and $D_t \wedge [a0]_0$, and remembering that the probabilities in (12.15) and (12.18) always contain the initial state $[a0]$ at $t = 0$ as an (implicit) condition. Also see the remarks in Sec. 14.4.)

Another family of consistent histories can be constructed in the following way. At the times $t = 1, 2$ use, in place of (12.7), a three-projector decomposition of the identity

$$I = [t\bar{a}] + [t\bar{b}] + J_t, \tag{12.19}$$

where the states $|t\bar{a}\rangle$, $|t\bar{b}\rangle$ are defined in (12.6), and

$$J_t = I - [t\bar{a}] - [t\bar{b}] = I - [tc] - [td] \tag{12.20}$$

is a projector for the particle to be someplace other than the two sites tc or td. At later times $t \geq 3$ use the decomposition (12.7). It is easy to show that in the case $f = 4$ the two histories

$$\begin{aligned} \bar{Y}^c &= [0a] \odot [1\bar{a}] \odot [2\bar{a}] \odot [3c] \odot [4c], \\ \bar{Y}^d &= [0a] \odot [1\bar{a}] \odot [2\bar{a}] \odot [3d] \odot [4d], \end{aligned} \tag{12.21}$$

each with weight 1/2, form the support of the sample space of a consistent family; all other histories have zero weight.

The histories $\bar{Y}^c$ and $\bar{Y}^d$ in (12.21) have the physical significance that at $t = 1$ and $t = 2$ the particle is in a coherent superposition of states in both output channels. After $t = 2$ a "split" occurs, and at later times the two histories are no longer identical: one represents the particle as traveling out the c channel, and the other the particle traveling out the d channel. What causes this split? To think of a physical cause for it is to look at the problem in the wrong way. Recall the case of a spin half particle with trivial dynamics, discussed in Sec. 9.3, with $S_z = 1/2$ initially and then $S_x = \pm 1/2$ at a later time. There is no physical transformation of the particle, since the dynamics is trivial. Instead, different aspects of the particle's spin angular momentum are being described at two successive times. In the same way, the histories in (12.21) allow us to describe a property at times $t = 1$ and $t = 2$, corresponding to the linear superposition $|m\bar{a}\rangle$, which cannot be described if we use the histories in (12.13). Conversely, using (12.21) makes it impossible to discuss whether the particle is in the c or in the d channel when $t = 1$ or 2, because these properties are incompatible with the projectors employed in $\bar{Y}^c$ and $\bar{Y}^d$. There is a similar split in the case of the histories Y^c and Y^d: they start with the same initial state $[0a]$, and the split occurs when t changes from 0 to 1. In this situation one may be tempted to suppose that the beam splitter causes the split, but that surely cannot be the case, for the very same beam splitter does not cause a split in the case of $\bar{Y}^c$ and $\bar{Y}^d$.

We have one family of histories based upon Y^c and Y^d, and a distinct family based upon $\bar{Y}^c$ and $\bar{Y}^d$. The two families are incompatible, as they have no common refinement. Which one provides the *correct* description of the physical system? Consider two histories of Great Britain: one a political history which discusses the monarchs, the other an intellectual history focusing upon developments in British science. Which is the *correct* history of Great Britain? That is not the proper way to compare them. Instead, there are certain questions which can be answered by one history rather than the other. For certain purposes one history is more useful, for other purposes the other is to be preferred. In the same way, both the Y^c, Y^d family and the $\bar{Y}^c, \bar{Y}^d$ family provide correct (stochastic) descriptions of the physical system, descriptions which are useful for answering different sorts of questions. There are, to be sure, certain questions which can be answered using either family, such as "Will the particle be in the c or the d channel at $t = 4$ if it was at $3c$ at $t = 3$?" For such questions, both families give precisely the same answer, in agreement with a general principle of consistency discussed in Sec. 16.3.

Next consider a family in which the histories start off like $\bar{Y}^c$ and $\bar{Y}^d$ in (12.21), but later on revert back to the coherent superposition states corresponding to (12.19); for example

$$\begin{aligned} Y' &= [0a] \odot [1\bar{a}] \odot [2\bar{a}] \odot [3c] \odot [4\bar{a}], \\ Y'' &= [0a] \odot [1\bar{a}] \odot [2\bar{a}] \odot [3d] \odot [4\bar{a}], \end{aligned} \tag{12.22}$$

plus other histories needed to make up a sample space. This family is not consistent. The reason is that the chain kets $|y'\rangle$ and $|y''\rangle$ corresponding to $K(Y')$ and $K(Y'')$ are non-zero multiples of $|4\bar{a}\rangle$, so $\langle y'|y''\rangle \neq 0$, and hence $K(Y')$ and $K(Y'')$ are not orthogonal to each other, see (11.28). There is a certain analogy between (12.22) and the inconsistent family for a spin half particle involving three times discussed in Sec. 10.3. The precise time at which the split and the rejoining occur is not important; for example, the chain operators associated with the histories

$$\begin{aligned} X' &= [0a] \odot [1c] \odot [2c] \odot [3c] \odot [4\bar{a}], \\ X'' &= [0a] \odot [1d] \odot [2d] \odot [3d] \odot [4\bar{a}] \end{aligned} \tag{12.23}$$

are also not mutually orthogonal, so the corresponding family is inconsistent. Inconsistency does not require a perfect rejoining; even a partial one can cause trouble! But why might someone want to consider families of histories of the form (12.22) or (12.23)? We will see in Ch. 13 that in the case of a simple interferometer the analogous histories look rather "natural", and it will be of some importance that they are not part of a consistent family.

12.2 Beam Splitter With Detector

Let us now add a detector of the sort described in Sec. 7.4 to the c output channel of the beam-splitter, Fig. 12.2. The detector has two states: $|0\hat{c}\rangle$ "ready", and $|1\hat{c}\rangle$ "triggered", which span a Hilbert space $\mathcal{C}$. The Hilbert space of the total quantum system is

$$\mathcal{H} = \mathcal{M} \otimes \mathcal{C}, \tag{12.24}$$

where $\mathcal{M}$ is the Hilbert space of the particle passing through the beam splitter, and the collection $\{|mz, n\hat{c}\rangle\}$ for different values of m, z, and n is an orthonormal basis of $\mathcal{H}$.

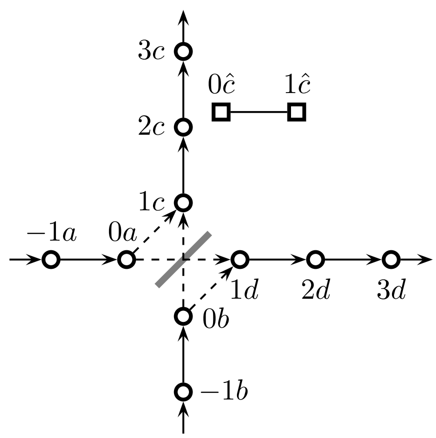

Figure 12.2: Toy beamsplitter with detector.

The unitary time development operator takes the form

$$T = S_b R_c, \tag{12.25}$$

where S_b is the unitary transformation defined in (12.1) and (12.2), extended in the usual way to the operator $S_b \otimes I$ on $\mathcal{M} \otimes \mathcal{C}$, and R_c (the subscript indicates that this detector is attached to the

c channel) is defined in analogy with (7.53) as

$$R_c|mz, n\hat{c}\rangle = |mz, n\hat{c}\rangle, \tag{12.26}$$

with the exception that

$$R_c|2c, n\hat{c}\rangle = |2c, (1-n)\hat{c}\rangle. \tag{12.27}$$

That is, R_c is the identity operator unless the particle is at the site $2c$, in which case the detector flips from $0\hat{c}$ to $1\hat{c}$, or $1\hat{c}$ to $0\hat{c}$. As noted in Sec. 7.4, such a detector does not perturb the motion of the particle, in the sense that the particle moves from $1c$ to $2c$ to $3c$, etc., at successive time steps whether or not the detector is present.

We shall assume an initial state

$$|\Psi_0\rangle = |0a, 0\hat{c}\rangle \tag{12.28}$$

at $t = 0$: the particle is at $0a$, about to enter the beam splitter, and the detector is ready. Unitary time development of this initial state leads to

$$|\Psi_t\rangle = T^t|\Psi_0\rangle = \begin{cases} \big(|tc\rangle + |td\rangle\big) \otimes |0\hat{c}\rangle/\sqrt{2} & \text{for } t = 1, 2, \\ \big(|tc, 1\hat{c}\rangle + |td, 0\hat{c}\rangle\big)/\sqrt{2} & \text{for } t \geq 3. \end{cases} \tag{12.29}$$

If one regards $|\Psi_t\rangle$ for $t \geq 3$ as representing a physical state or physical property of the combined particle and detector, then the detector is not in a definite state. Instead one has a toy counterpart of a *macroscopic quantum superposition* (MQS) or *Schrödinger's cat* state. See the discussion in Sec. 9.6. It is impossible to say whether or not the detector has detected something at times $t \geq 3$ if one uses a unitary family based upon the initial state $|\Psi_0\rangle$.

A useful family of histories for studying the process of detection is based on the initial state $|\Psi_0\rangle$ and a decomposition of the identity in pure states

$$I = \sum_{m,z,n} [mz, n\hat{c}], \tag{12.30}$$

in which the particle has a definite location and the detector is in one of its pointer states at every time $t > 0$. The histories

$$\begin{aligned} Z^c &= [0a, 0\hat{c}] \odot [1c, 0\hat{c}] \odot [2c, 0\hat{c}] \odot [3c, 1\hat{c}] \odot [4c, 1\hat{c}] \cdots, \\ Z^d &= [0a, 0\hat{c}] \odot [1d, 0\hat{c}] \odot [2d, 0\hat{c}] \odot [3d, 0\hat{c}] \odot [4d, 0\hat{c}] \cdots, \end{aligned} \tag{12.31}$$

continuing for as long a sequence of times as one wants to consider, are the obvious counterparts of Y^c and Y^d in (12.13). Because the final projectors are orthogonal, $K(Z^c)$ and $K(Z^d)$ are orthogonal, and it is not hard to show that Z^c and Z^d constitute the support of a consistent family $\mathcal{F}$ based on the initial state $|\Psi_0\rangle$. The physical interpretation of these histories is straightforward. In Z^c the particle moves out the c channel and triggers the detector, changing $0\hat{c}$ to $1\hat{c}$ as it moves from $2c$ to $3c$. In Z^d the particle moves out the d channel, and the detector remains in its untriggered or ready state $0\hat{c}$.

We can use the property that the detector has (or has not) detected the particle at some time $t' \geq 3$ to determine which channel the particle is in, by computing a conditional probability. Thus one finds—see the discussion following (12.15)—that

$$\begin{aligned} &\Pr(C_t \,|\, [1\hat{c}]_{t'}) = 1, \quad \Pr(D_t \,|\, [1\hat{c}]_{t'}) = 0, \\ &\Pr(C_t \,|\, [0\hat{c}]_{t'}) = 0, \quad \Pr(D_t \,|\, [0\hat{c}]_{t'}) = 1, \end{aligned} \tag{12.32}$$

for $t' \geq 3$ and $t \geq 1$. That is, if at some time $t' \geq 3$, the detector has detected the particle, then at time t, the particle is (or was) in the c and not in the d channel, while if the detector has not detected the particle, the particle is (or was) in the d and not in the c channel.

Note that the conditional probabilities in (12.32) are valid not simply for $t \geq 3$; they also hold for $t = 1$ and 2. That is, if the detector is triggered at time $t' = 3$, then the particle was in the c channel at $t = 1$ and 2, and if the detector is not triggered at $t' = 3$, then at these earlier times the particle was in the d channel. These results are perfectly reasonable from a physical point of view. How could the particle have triggered the detector unless it was already moving out along the c channel? And if it did not trigger the detector, where could it have been except in the d channel? As long as the particle does not hop from one channel to the other in some magical way, the results in (12.32) are just what one would expect.

Another family in which the detector is always in one of its pointer states is the counterpart of (12.21), modified by the addition of a detector:

$$\begin{aligned} \bar{Z}^c &= [0a, 0\hat{c}] \odot [1\bar{a}, 0\hat{c}] \odot [2\bar{a}, 0\hat{c}] \odot [3c, 1\hat{c}] \odot [4c, 1\hat{c}] \cdots , \\ \bar{Z}^d &= [0a, 0\hat{c}] \odot [1\bar{a}, 0\hat{c}] \odot [2\bar{a}, 0\hat{c}] \odot [3d, 0\hat{c}] \odot [4d, 0\hat{c}] \cdots . \end{aligned} \tag{12.33}$$

The chain operators for $\bar{Z}^c$ and $\bar{Z}^d$ are orthogonal, and it is easy to find zero-weight histories to complete the sample space, so that (12.33) is the support of a consistent family $\mathcal{G}$. It differs from $\mathcal{F}$, (12.31), in that at $t = 1$ and 2 the particle is in the superposition state $|t\bar{a}\rangle$ rather than in the c or the d channel, but for times after $t = 2$ $\mathcal{F}$ and $\mathcal{G}$ are identical.

Both families $\mathcal{F}$, (12.31), and $\mathcal{G}$, (12.33), represent equally good quantum descriptions. The only difference is that they allow one to discuss somewhat different properties of the particle at a time after it has passed through the beam splitter and before it has been detected. In particular, if one is interested in knowing the location of the particle before the measurement occurred (or could have occurred), it is necessary to employ a consistent family in which questions about its location are meaningful, so $\mathcal{F}$ must be used, not $\mathcal{G}$. On the other hand, if one is interested in whether the particle was in the superposition $|1\bar{a}\rangle$ at $t = 1$ rather than in $|1\bar{b}\rangle$—see the definitions in (12.6)—then it is necessary to use $\mathcal{G}$, for questions related to such superpositions are meaningless in $\mathcal{F}$.

The family $\mathcal{G}$, (12.33), is useful for understanding the idea, which goes back to von Neumann, that a measurement produces a "collapse" or "reduction" of the wave function. As applied to our toy model, a measurement which serves to detect the presence of the particle in the c channel is thought of as collapsing the superposition wave function $|2\bar{a}\rangle$ produced by unitary time evolution into a state $|3c\rangle$ located in the c channel. This is the step from $[2\bar{a}, 0\hat{c}]$ to $[3c, 1\hat{c}]$ in the history $\bar{Z}^c$. Similarly, if the detector does not detect the particle, $|2\bar{a}\rangle$ collapses to a state $|3d\rangle$ in the d channel, as represented by the step from $t = 2$ to $t = 3$ in the history $\bar{Z}^d$.

The approach to measurements based on wave function collapse is the subject of Sec. 18.2. While it can often be employed in a way which gives correct results, wave function collapse is not really needed, since the same results can always be obtained by straightforward use of conditional probabilities. On the other hand, it has given rise to a lot of confusion, principally because the collapse tends to be thought of as a physical effect produced by the measuring apparatus. With reference to our toy model, this might be a reasonable point of view when the particle is detected to be in the c channel, but it seems very odd that a *failure* to detect the particle in the c channel has the effect of collapsing its wave function into the d channel, which might be a long ways away from the c detector. That the collapse is not any sort of physical effect is clear from the fact that it occurs in the family (12.21) in the absence of a detector, and in $\mathcal{F}$, (12.31), it occurs prior to detection. To be sure, in $\mathcal{F}$ one might suppose that the collapse is caused by the beam splitter. However, one could modify (12.31) in an obvious way to produce a consistent family in which the collapse takes place between $t = 1$ and $t = 2$, and thus has nothing to do with either the beam splitter or detector.

Another way in which the collapse approach to quantum measurements is somewhat unsatisfactory is that it does not provide a connection between the outcome of a measurement and a corresponding property of the measured system before the measurement took place. For example, if at $t \geq 3$ the detector is in the state $1\hat{c}$, there is no way to infer that the particle was earlier in the c channel if one uses the family (12.33) rather than (12.31). The connection between measurements and what they measure will be discussed in Ch. 17.

12.3 Time-Elapse Detector

A simple two-state toy detector is useful for thinking about a number of situations in quantum theory involving detection and measurement. However, it has its limitations. In particular, unlike real detectors, it does not have sufficient complexity to allow the *time* at which an event occurs to be recorded by the detector. While it is certainly possible to include a clock as part of a toy detector, a slightly simpler solution to the timing problem is to use a *time-elapse detector*: when an event is detected, a clock is started, and reading this clock tells how much time has elapsed since the detection occurred. As in Sec. 7.4, the Hilbert space $\mathcal{H}$ is a tensor product $\mathcal{M} \otimes \mathcal{N}$ of the space $\mathcal{M}$ of the particle, spanned by kets $|m\rangle$ with $-M_a \leq m \leq M_b$, and the space $\mathcal{N}$ of the detector, with kets $|n\rangle$ labeled by n in the range

$$-N \leq n \leq N. \tag{12.34}$$

In effect, one can think of the detector as a second particle that moves according to an appropriate dynamics. However, to avoid confusion the term *particle* will be reserved for the toy particle whose position is labeled by m, and which the detector is designed to detect, while n will be the position of the detector's *pointer* (see the remarks at the end of Sec. 9.5). We shall suppose that M_a, M_b, and N are sufficiently large that we do not have to worry about either the particle or the pointer "coming around the cycle" during the time period of interest.

The unitary time development operator is

$$T = SRS_d, \tag{12.35}$$

where S is the shift operator on $\mathcal{M}$,

$$S|m\rangle = |m+1\rangle, \tag{12.36}$$

with a periodic boundary condition $S|M_b\rangle = |-M_a\rangle$, and S_d acts on $\mathcal{N}$,

$$S_d|n\rangle = |n+1\rangle, \tag{12.37}$$

with the exceptions

$$S_d|0\rangle = |0\rangle, \quad S_d|-1\rangle = |1\rangle, \tag{12.38}$$

and $S_d|N\rangle = |-N\rangle$ to take care of the periodic boundary condition. The unitary operator R which couples the pointer to the particle is the identity,

$$R|m,n\rangle = |m,n\rangle, \tag{12.39}$$

except for

$$R|2,0\rangle = |2,1\rangle, \quad R|2,1\rangle = |2,0\rangle. \tag{12.40}$$

That is, when the particle is at $m = 2$, R moves the pointer from $n = 0$ to $n = 1$, or from $n = 1$ to $n = 0$, while if the pointer is someplace else, R has no effect on it. The unitarity of T in (12.35) follows from that of S, R, and S_d.

When its pointer is at $n = 0$, the detector is in its "ready" state, where it remains until the particle reaches $m = 2$, at which point the "detection event" (12.40) occurs, and the pointer hops to $n = 1$ at the same time as the particle hops to $m = 3$, since T includes the shift operator S for the particle, (12.35). This is identical to the operation of the two-state detector of Sec. 7.4. But once the detector pointer is at $n = 1$ it keeps going, (12.37), so a typical unitary time development of $|m,n\rangle$ is of the form

$$|0,0\rangle \mapsto |1,0\rangle \mapsto |2,0\rangle \mapsto |3,1\rangle \mapsto |4,2\rangle \mapsto |5,3\rangle \mapsto \cdots. \tag{12.41}$$

Thus the pointer reading n (assumed to be less than N) tells how much time has elapsed since the detection event occurred.

As an example of the operation of this detector in a stochastic context, suppose that at $t = 0$ there is an initial state

$$|\Psi_0\rangle = |\psi_0\rangle \otimes |0\rangle, \tag{12.42}$$

where the particle wave packet

$$|\psi_0\rangle = a|0\rangle + b|1\rangle + c|2\rangle \tag{12.43}$$

has three non-zero coefficients a, b, c. Consider histories which for $t > 0$ employ a decomposition of the identity corresponding to the orthonormal basis $\{|m,n\rangle\}$. The chain operators for the three histories

$$\begin{aligned} Z^0 &= [\Psi_0] \odot [1,0] \odot [2,0] \odot [3,1], \\ Z^1 &= [\Psi_0] \odot [2,0] \odot [3,1] \odot [4,2], \\ Z^2 &= [\Psi_0] \odot [3,1] \odot [4,2] \odot [5,3], \end{aligned} \tag{12.44}$$

involving the four times $t = 0, 1, 2, 3$, are obviously orthogonal to one another (because of the final projectors, Sec. 11.3). The corresponding weights are $|a|^2$, $|b|^2$, and $|c|^2$, while all other histories beginning with $[\Psi_0]$ have zero weight. Hence (12.44) is the support of a consistent family with initial state $|\Psi_0\rangle$.

Suppose that the pointer is located at $n = 2$ when $t = 3$. Since the pointer position indicates the time that has elapsed since the particle was detected, we should be able to infer that the detection event $[2,0]$ occurred at $t = 3 - 2 = 1$. Indeed, one can show that

$$\Pr([2,0] \text{ at } t = 1 \,|\, n = 2 \text{ at } t = 3) = 1, \tag{12.45}$$

using the fact that the condition $n = 2$ when $t = 3$ is only true for Z^1. If the pointer is at $n = 1$ when $t = 3$, one can use the family (12.44) to show not only that the detection event $[2, 0]$ occurred at $t = 2$, but also that at $t = 1$ the particle was at $m = 1$, one site to the left of the detector. Being able to infer where the particle was before it was detected is intuitively reasonable, and is the sort of inference often employed when analyzing data from real detectors in the laboratory. Such inferences depend, of course, on using an appropriate consistent family, as discussed in Sec. 12.2.

12.4 Toy Alpha Decay

A toy model of alpha decay was introduced in Sec. 7.4, and discussed using the Born rule in Sec. 9.5. We assume the sites are labeled as in Fig. 7.2 on page 91, and will employ the same $T = S_a$ dynamics used previously, (7.56). That is,

$$S_a|m\rangle = |m+1\rangle, \tag{12.46}$$

with the exceptions

$$S_a|0\rangle = \alpha|0\rangle + \beta|1\rangle, \quad S_a|-1\rangle = \gamma|0\rangle + \delta|1\rangle, \tag{12.47}$$

together with a periodic boundary condition. The coefficients α, β, γ, and δ satisfy (7.58).

Consider histories which begin with the initial state

$$|\psi_0\rangle = |0\rangle, \tag{12.48}$$

the alpha particle inside the nucleus, and employ a decomposition of the identity based upon particle position states $|m\rangle$ at all later times. That such a family of histories, thought of as extending from the initial state at $t = 0$ till a later time $t = f$, is consistent can be seen by working out what happens when f is small. In particular, when $f = 1$, there are two histories with non-zero weight:

$$\begin{aligned} &[0] \odot [0], \\ &[0] \odot [1]. \end{aligned} \tag{12.49}$$

The chain operators are orthogonal because the projectors at the final time are mutually orthogonal (Sec. 11.3). With $f = 2$, there are three histories with non-zero weight:

$$\begin{aligned} &[0] \odot [0] \odot [0], \\ &[0] \odot [0] \odot [1], \\ &[0] \odot [1] \odot [2] \end{aligned} \tag{12.50}$$

and again it is obvious that the chain operators are orthogonal, so that the corresponding family is consistent.

These examples suggest the general pattern, valid for any f. The support of the consistent family contains a history in which $m = 0$ at all times, together with histories with a decay time $t = \tau$, with τ in the range $0 \leq \tau \leq f - 1$, of the form

$$[0]_0 \odot [0]_1 \odot \cdots [0]_\tau \odot [1]_{\tau+1} \odot [2]_{\tau+2} \odot \cdots . \tag{12.51}$$

That is, the alpha particle remains in the nucleus, $m = 0$, until the time $t = \tau$, then hops to $m = 1$ at $t = \tau + 1$, and after that it keeps going. If one uses this particular family of histories, the quantum problem is much the same as that of a classical particle which hops out of a well with a certain probability at each time step, and once out of the well moves away from it at a constant speed. This is not surprising, since as long as one employs a single consistent family the mathematics of a quantum stochastic process is formally identical to that of a classical stochastic process.

In Sec. 9.5 a simple two-state detector was used in analyzing toy alpha decay by means of the Born rule. Additional insight can be gained by replacing the two-state detector in Fig. 9.1 with the time-elapse detector of Sec. 12.3 to detect the alpha particle as it hops from $m = 2$ to $m = 3$ after leaving the nucleus. On the Hilbert space $\mathcal{M} \otimes \mathcal{N}$ of the alpha particle and detector pointer, the unitary time development operator is

$$T = S_a R S_d, \tag{12.52}$$

where S_d and R are defined in (12.37) to (12.40).

Suppose that at the time $t = \bar{t}$ the detector pointer is at $\bar{n}$. Then the detection event should have occurred at the time $\bar{t} - \bar{n}$. And since the particle was detected at the site $m = 2$, the actual decay time τ when it left the nucleus would have been a bit earlier,

$$\tau = \bar{t} - \bar{n} - 2, \tag{12.53}$$

because of the finite travel time from the nucleus to the detector. This line of reasoning can be confirmed by a straightforward calculation of the conditional probabilities

$$\begin{aligned} \Pr(m = 0 \text{ at } t = \bar{t} - \bar{n} - 2 \,|\, n = \bar{n} \text{ at } t = \bar{t}) = 1, \\ \Pr(m = 0 \text{ at } t = \bar{t} - \bar{n} - 1 \,|\, n = \bar{n} \text{ at } t = \bar{t}) = 0. \end{aligned} \tag{12.54}$$

That is, at the time τ given in (12.53), the particle was still in the nucleus, while one time step later it was no longer there. (Of course this only makes sense if $\bar{t}$ and $\bar{n}$ are such that $\Pr(n = \bar{n} \text{ at } t = \bar{t})$ is positive.) Note once again that by adopting an appropriate family of histories one can make physically reasonable inferences about events prior to the detection of the alpha particle.

Does the fact that we can assign a decay time in the case of our toy model mean that the same thing is possible for real alpha decay? The answer is presumably "yes", provided one does not require that the decay time be defined too precisely. However, finding a suitable criterion for the nucleus to have or have not decayed and checking consistency conditions for an appropriate family pose non-trivial technical issues, and the matter does not seem to have been studied in detail. Note that even in the toy model the decay time is not precisely defined, because time is discretized, and $\tau + 1$ has as much justification for being identified with the decay time as does τ. This uncertainty can, however, be much shorter than the half life of the nucleus, which is of the order of $|\beta|^{-2}$.