Chapter 8

Stochastic Histories

8.1 Introduction

Despite the fact that classical mechanics employs deterministic dynamical laws, random dynamical processes often arise in classical physics, as well as in everyday life. A *stochastic* or *random* process is one in which states-of-affairs at successive times are not related to one another by deterministic laws, and instead probability theory is employed to describe whatever regularities exist. Tossing a coin or rolling a die several times in succession are examples of stochastic processes in which the previous history is of very little help in predicting what will happen in the future. The motion of a baseball is an example of a stochastic process which is to some degree predictable using classical equations of motion that relate its acceleration to the total force acting upon it. However, a lack of information about its initial state (e.g., whether it is spinning), its precise shape, and the condition and motion of the air through which it moves limits the precision with which one can predict its trajectory.

The Brownian motion of a small particle suspended in a fluid and subject to random bombardment by the surrounding molecules of fluid is a well-studied example of a stochastic process in classical physics. Whereas the instantaneous velocity of the particle is hard to predict, there is a probabilistic correlation between successive positions, which can be predicted using stochastic dynamics and checked by experimental measurements. In particular, given the particle's position at a time t, it is possible to compute the probability that it will have moved a certain distance by the time $t + \Delta t$. The stochastic description of the motion of a Brownian particle uses the deterministic law for the motion of an object in a viscous fluid, and assumes that there is, in addition, a random force or "noise" which is unpredictable, but whose statistical properties are known.

In classical physics the need to use stochastic rather than deterministic dynamical processes can be blamed on ignorance. If one knew the precise positions and velocities of all the molecules making up the fluid in which the Brownian particle is suspended, along with the same quantities for the molecules in the walls of the container and inside the Brownian particle itself, it would in principle be possible to integrate the classical equations of motion and make precise predictions about the motion of the particle. Of course, integrating the classical equations of motion with infinite precision is not possible. Nonetheless, in classical physics one can, in principle, construct more and more refined descriptions of a mechanical system, and thereby continue to reduce the noise in the stochastic dynamics in order to come arbitrarily close to a deterministic description. Knowing the spin imparted to a baseball by the pitcher allows a more precise prediction of its future trajectory. Knowing the positions and velocities of the fluid molecules inside a sphere centered at a Brownian particle makes it possible to improve one's prediction of its motion, at least over a short time interval.

The situation in quantum physics is similar, up to a point. A quantum description can be made more precise by using smaller, i.e., lower-dimensional subspaces of the Hilbert space. However, while the refinement of a classical description can go on indefinitely, one reaches a limit in the quantum case when the subspaces are one-dimensional, since no finer description is possible. However, at this level quantum dynamics is still stochastic: there is an irreducible "quantum noise" which cannot be eliminated, even in principle. To be sure, quantum theory allows for a deterministic (and thus noise free) unitary dynamics, as discussed in the previous chapter. But there are many processes in the real world which cannot be discussed in terms of purely unitary dynamics based upon Schrödinger's equation. Consequently, stochastic descriptions are a fundamental part of quantum mechanics in a sense which is not true in classical mechanics.

In this chapter we focus on the kinematical aspects of classical and quantum stochastic dynamics: how to construct sample spaces and the corresponding event algebras. As usual, classical dynamics is simpler and provides a valuable guide and useful analogies for the quantum case, so various classical examples are taken up in Sec. 8.2. Quantum dynamics is the subject of the remainder of the chapter.

8.2 Classical Histories

Consider a coin which is tossed three times in a row. The eight possible outcomes of this experiment are HHH, HHT, HTH, $\dots TTT$: heads on all three tosses, heads the first two times and tails the third, and so forth. These eight possibilities constitute a *sample space* as that term is used in probability theory, see Sec. 5.1, since the different possibilities are mutually exclusive, and one and only one of them will occur in any particular experiment in which a coin is tossed three times in a row. The *event algebra* (Sec. 5.1) consists of the 2^8 subsets of elements in the sample space: the empty set, HHH by itself, the pair $\{HHT, TTT\}$, and so forth. The elements of the sample space will be referred to as *histories*, where a history is to be thought of as a *sequence of events at successive times*. Members of the event algebra will also be called "histories" in a somewhat looser sense, or *compound histories* if they include two or more elements from the sample space.

As a second example, consider a die which is rolled f times in succession. The sample space consists of 6^f possibilities $\{s_1, s_2, \dots, s_f\}$, where each s_j takes some value between 1 and 6.

A third example is a Brownian particle moving in a fluid and observed under a microscope at successive times t_1, t_2, $\dots t_f$. The sequence of positions $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\dots \boldsymbol{r}_f$ is an example of a history, and the sample space consists of all possible sequences of this type. Since any measuring instrument has finite resolution, one can, if one wants, suppose that for the purpose of recording the data the region inside the fluid is thought of as divided up into a collection of small cubical cells, with $\boldsymbol{r}_j$ the label of the cell containing the particle at time t_j.

A fourth example is a particle undergoing a random walk in one dimension, a sort of "toy model" of Brownian motion. Assume that the location of the particle or random walker, denoted by s, is an integer in the range

$$-M_a \leq s \leq M_b. \tag{8.1}$$

One could allow s to be any integer, but using the limited range (8.1) results in a finite sample space of $M = M_a + M_b + 1$ possibilities at any given time. At each time step the particle either remains where it is, or hops to the right or to the left. Hence a *history* of the particle's motion consists in giving its positions at a set of times $t = 0, 1, \ldots f$ as a sequence of integers

$$\boldsymbol{s} = (s_0, s_1, s_2, \ldots s_f), \tag{8.2}$$

where each s_j falls in the interval (8.1). The *sample space of histories* consists of the M^{f+1} different sequences $\boldsymbol{s}$. (Letting s_0 rather than s_1 be the initial position of the particle is of no importance; the convention used here agrees with that in the next chapter.) One could employ histories extending to $t = \infty$, but that would mean using an infinite sample space.

This sample space can be thought of as produced by successively refining an initial, coarse sample space in which s_0 takes one of M possible values, and nothing is said about what happens at later times. Histories involving the two times $t = 0$ and 1 are produced by taking a point in this initial sample space, say $s_0 = 3$, and "splitting it up" into two-time histories of the form $(3, s_1)$, where s_1 can take on any one of the M values in (8.1). Given a point, say $(3, 2)$, in this new sample space, it can again be split up into elements of the form $(3, 2, s_2)$, and so forth. Note that any history involving less than $n + 1$ times can be thought of as a compound history on the full sample space. Thus $(3, 2)$ consists of all sequences $\boldsymbol{s}$ for which $s_0 = 3$ and $s_1 = 2$. Rather than starting with a coarse sample space of events at $t = 0$, one could equally well begin with a later time, such as all the possibilities for s_2 at $t = 2$, and then refine this space by including additional details at both earlier and later times.

8.3 Quantum Histories

A *quantum history* of a physical system is a sequence of *quantum events* at successive times, where a quantum event at a particular time can be any *quantum property* of the system in question. Thus given a set of times $t_1 < t_2 < \cdots t_f$, a quantum history is specified by a collection of projectors $(F_1, F_2, \ldots F_f)$, one projector for each time. It is convenient, both for technical and for conceptual reasons, to suppose that the number f of distinct times is finite, though it might be very large. It is always possible to add additional times to those in the list $t_1 < t_2 < \cdots t_f$ in the manner indicated in Sec. 8.4. Sometimes the initial time will be denoted by t_0 rather than t_1.

For a spin-half particle, $([z^+],\ [x^+])$ is an example of a history involving 2 times, while $([z^+],\ [x^+],\ [z^+])$ is an example involving 3 times.

As a second example, consider a harmonic oscillator. A possible history with 3 different times is the sequence of events

$$F_1 = [\phi_1] + [\phi_2], \quad F_2 = [\phi_1], \quad F_3 = X, \tag{8.3}$$

where $[\phi_n]$ is the projector on the energy eigenstate with energy $(n+1/2)\hbar\omega$, and X is the projector defined in (4.20) corresponding to the position x lying in the interval $x_1 \leq x \leq x_2$. Note that the projectors making up a history do not have to project onto a one-dimensional subspace of the Hilbert space. In this example, F_1 projects onto a two-dimensional subspace, F_2 onto a one-dimensional subspace, and X onto an infinite-dimensional subspace.

As a third example, consider a coin tossed three times in a row. A physical coin is made up of atoms, so it has in principle a (rather complicated) quantum mechanical description. Thus a "classical" property such as "heads" will correspond to some quantum projector H onto a subspace of enormous dimension, and there will be another projector T for "tails". Then by using the projectors

$$F_1 = H, \quad F_2 = T, \quad F_3 = T \tag{8.4}$$

at successive times one obtains a quantum history HTT for the coin.

As a fourth example of a quantum history, consider a Brownian particle suspended in a fluid. Whereas this is usually described in classical terms, the particle and the surrounding fluid are, in reality, a quantum system. At time t_j let F_j be the projector, in an appropriate Hilbert space, for the property that the center of mass of the Brownian particle is inside a particular cubical cell. Then $(F_1, F_2, \ldots F_f)$ is the quantum counterpart of the classical history $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\ldots \boldsymbol{r}_f$ introduced earlier, with $\boldsymbol{r}_j$ understood as a cell label, rather than a precise position.

One does not normally think of coin tossing in "quantum" terms, and there is really no advantage to doing so, since a classical description is simpler, and is perfectly adequate. Similarly, a classical description of the motion of a Brownian particle is usually quite adequate. However, these examples illustrate the fact that the concept of a quantum history is really quite general, and is by no means limited to processes and events at an atomic scale, even though that is where quantum histories are most useful, precisely because the corresponding classical descriptions are not adequate.

The sample space of a coin tossed f times in a row is formally the same as the sample space of f coins tossed simultaneously: each consists of 2^f mutually exclusive possibilities. Since in quantum theory the Hilbert space of a collection of f systems is the tensor product of the separate Hilbert spaces, Ch. 6, it seems reasonable to use a tensor product of f spaces for describing the different histories of a single quantum system at f successive times. Thus we define a *history Hilbert space* as a tensor product

$$\breve{\mathcal{H}} = \mathcal{H}_1 \odot \mathcal{H}_2 \odot \cdots \mathcal{H}_f, \tag{8.5}$$

where for each j, $\mathcal{H}_j$ is a copy of the Hilbert space $\mathcal{H}$ used to describe the system at a single time, and $\odot$ is a variant of the tensor product symbol $\otimes$. We could equally well write $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots$, but it is helpful to have a distinctive notation for a tensor product when the factors in it refer to different times, and reserve $\otimes$ for a tensor product of spaces at a single time. On the space $\breve{\mathcal{H}}$ the history $(F_1, F_2, \ldots F_f)$ is represented by the (tensor) product projector

$$Y = F_1 \odot F_2 \odot \cdots F_f. \tag{8.6}$$

That Y is a projector, that is, $Y^\dagger = Y = Y^2$, follows from the fact that each F_j is a projector, and from the rules for adjoints and products of operators on tensor products as discussed in Sec. 6.4.

8.4 Extensions and Logical Operations on Histories

Suppose that $f = 3$ in (8.6), so that

$$Y = F_1 \odot F_2 \odot F_3. \tag{8.7}$$

This history can be *extended* to additional times by introducing the identity operator at the times not included in the initial set t_1, t_2, t_3. Suppose, for example, that we wish to add an additional time t_4 later than t_3. Then for times $t_1 < t_2 < t_3 < t_4$, (8.7) is equivalent to

$$Y = F_1 \odot F_2 \odot F_3 \odot I, \tag{8.8}$$

because the identity operator I represents the property which is always true, and therefore provides no additional information about the system at t_4. In the same way, one can introduce earlier and intermediate times, say t_0 and $t_{1.5}$, in which case (8.7) is equivalent to

$$Y = I \odot F_1 \odot I \odot F_2 \odot F_3 \odot I \tag{8.9}$$

on the history space $\breve{\mathcal{H}}$ for the times $t_0 < t_1 < t_{1.5} < t_2 < t_3 < t_4$. We shall always use a notation in which the events in a history are in temporal order, with time increasing from left to right.

The notational convention for extensions of operators introduced in Sec. 6.4 justifies using the same symbol Y in (8.7), (8.8) and (8.9). And its intuitive significance is precisely the same in all three cases: Y means "F_1 at t_1, F_2 at t_2, and F_3 at t_3", and tells us nothing at all about what is happening at any other time. Using the same symbol for F and $F \odot I$ can sometimes be confusing for the reason pointed out at the end of Sec. 6.4. For example, the projector for a two-time history of a spin-half particle can be written as an operator product

$$[z^+] \odot [x^+] = \big([z^+] \odot I\big) \cdot \big(I \odot [x^+]\big) \tag{8.10}$$

of two projectors. If on the right side we replace $([z^+] \odot I)$ with $[z^+]$ and $(I \odot [x^+])$ with $[x^+]$, the result $[z^+] \cdot [x^+]$ is likely to be incorrectly interpreted as the product of two non-commuting operators on a single copy of the Hilbert space $\mathcal{H}$, rather than as the product of two commuting operators on the tensor product $\mathcal{H}_1 \odot \mathcal{H}_2$. Using the longer $([z^+] \odot I)$ avoids this confusion.

If histories are written as projectors on the history Hilbert space $\breve{\mathcal{H}}$, the rules for the logical operations of negation, conjunction, and disjunction are precisely the same as for quantum properties at a single time, as discussed in Secs. 4.4 and 4.5. In particular, the negation of the history Y, "Y did not occur", corresponds to a projector

$$\tilde{Y} = \breve{I} - Y, \tag{8.11}$$

where $\breve{I}$ is the identity on $\breve{\mathcal{H}}$. (Our notational convention allows us to write $\breve{I}$ as I, but $\breve{I}$ is clearer.)

Note that a history does not occur if any event in it fails to occur. Thus the negation of HH when a coin is tossed two times in a row is not TT, but instead the compound history consisting of HT, TH, and TT. Similarly, the negation of the quantum history

$$Y = F_1 \odot F_2 \tag{8.12}$$

given by (8.11) is a sum of three orthogonal projectors,

$$\tilde{Y} = F_1 \odot \tilde{F}_2 + \tilde{F}_1 \odot F_2 + \tilde{F}_1 \odot \tilde{F}_2, \tag{8.13}$$

where $\tilde{F}_j$ means $I - F_j$. Note that the compound history $\tilde{Y}$ in (8.13) cannot be written in the form $G_1 \odot G_2$, that is, as an event at t_1 followed by another event at t_2.

The conjunction Y AND Y', or $Y \wedge Y'$, of two histories is represented by the product YY' of the projectors, provided they commute with each other. If $YY' \neq Y'Y$, the conjunction is not defined. The situation is thus entirely analogous to the conjunction of two quantum properties at a single time, as discussed in Secs. 4.5 and 4.6. Let us suppose that the history

$$Y' = F_1' \odot F_2' \odot F_3' \tag{8.14}$$

is defined at the same three times as Y in (8.7). Their conjunction is represented by the projector

$$Y' \wedge Y = Y'Y = F_1'F_1 \odot F_2'F_2 \odot F_3'F_3, \tag{8.15}$$

which is equal to YY' provided that at each of the three times the projectors in the two histories commute:

$$F_j'F_j = F_jF_j' \text{ for } j = 1, 2, 3. \tag{8.16}$$

However, there is a case in which Y and Y' commute even if some of the conditions in (8.16) are not satisfied. It occurs when the product of the two projectors at one of the times is zero, for this means that $YY' = 0$ independent of what projectors occur at other times. Here is an example involving a spin-half particle:

$$\begin{aligned} Y &= [x^+] \odot [x^+] \odot [z^+], \\ Y' &= [y^+] \odot [z^+] \odot [z^-]. \end{aligned} \tag{8.17}$$

The two projectors at t_1, $[x^+]$ and $[y^+]$, clearly do not commute with each other, and the same is true at time t_2. However, the projectors at t_3 are orthogonal, and thus $YY' = 0 = Y'Y$.

A simple example of a non-vanishing conjunction is provided by a spin-half particle and two histories

$$Y = [z^+] \odot I, \quad Y' = I \odot [x^+], \tag{8.18}$$

defined at the times t_1 and t_2. The conjunction is

$$Y' \wedge Y = Y'Y = YY' = [z^+] \odot [x^+], \tag{8.19}$$

and this is sensible, for the intuitive significance of (8.19) is "$S_z = +1/2$ at t_1 and $S_x = +1/2$ at t_2." Indeed, any history of the form (8.6) can be understood as "F_1 at t_1, and F_2 at t_2, and $\ldots F_f$ at t_f." This example also shows how to generate the conjunction of two histories defined at different sets of times. First one must extend each history by including I at additional times until the extended histories are defined on a common set of times. If the extended projectors commute with each other, the operator product of the projectors, as in (8.15), is the projector for the conjunction of the two histories.

The disjunction "Y' or Y or both" of two histories is represented by a projector

$$Y' \vee Y = Y' + Y - Y'Y \tag{8.20}$$

provided $Y'Y = YY'$; otherwise it is undefined. The intuitive significance of the disjunction of two (possibly compound) histories is what one would expect, though there is a subtlety associated with the quantum disjunction which does not arise in the case of classical histories, as has already been noted in Sec. 4.5 for the case of properties at a single time. It can best be illustrated by means of an explicit example. For a spin-half particle, define the two histories

$$Y = [z^+] \odot [x^+], \quad Y' = [z^+] \odot [x^-]. \tag{8.21}$$

The projector for the disjunction is

$$Y \vee Y' = Y + Y' = [z^+] \odot I, \tag{8.22}$$

since in this case $YY' = 0$. The *projector* $Y \vee Y'$ tells us nothing at all about the spin of the particle at the second time: in and of itself it does *not* imply that $S_x = +1/2$ or $S_x = -1/2$ at t_2, since the subspace of $\breve{\mathcal{H}}$ on which it projects contains, among others, the history $[z^+] \odot [y^+]$, which is incompatible with S_x having any value at all at t_2. On the other hand, when the projector $Y \vee Y'$ occurs in the context of a discussion in which both Y and Y' make sense, it can be safely interpreted as meaning (or implying) that at t_2 either $S_x = +1/2$ or $S_x = -1/2$, since any other possibility, such as $S_y = +1/2$, would be incompatible with Y and Y'.

This example illustrates an important principle of quantum reasoning: The *context*, that is, the sample space or event algebra used for constructing a quantum description or discussing the histories of a quantum system, can make a difference in how one understands or interprets various symbols. In quantum theory it is important to be clear about precisely what sample space is being used.

8.5 Sample Spaces and Families of Histories

As discussed in Sec. 5.2, a sample space for a quantum system at a single time is a decomposition of the identity operator for the Hilbert space $\mathcal{H}$: a collection of mutually orthogonal projectors which sum to I. In the same way, a sample space of histories is a decomposition of the identity on the history Hilbert space $\breve{\mathcal{H}}$, a collection $\{Y^\alpha\}$ of mutually orthogonal projectors representing histories which sum to the history identity:

$$\breve{I} = \sum_\alpha Y^\alpha. \tag{8.23}$$

It is convenient to label the history projectors with a superscript in order to be able to reserve the subscript position for time. Since the square of a projector is equal to itself, we will not need to use superscripts on projectors as exponents.

Associated with a sample space of histories is a Boolean "event" algebra, called a *family of histories*, consisting of projectors of the form

$$Y = \sum_\alpha \pi^\alpha Y^\alpha, \tag{8.24}$$

with each π^α equal to 0 or 1, as in (5.12). Histories which are members of the sample space will be called *elementary* histories, whereas those of the form (8.24) with two or more π^α equal to 1 are *compound* histories. The term "family of histories" is also used to denote the sample space of histories which generates a particular Boolean algebra. Given the intimate connection between the sample space and the corresponding algebra, this double usage is unlikely to cause confusion.

The simplest way to introduce a history sample space is to use a *product of sample spaces* as that term was defined in Sec. 6.6. Assume that at each time t_j there is a decomposition of the identity I_j for the Hilbert space $\mathcal{H}_j$,

$$I_j = \sum_{\alpha_j} P_j^{\alpha_j}, \tag{8.25}$$

where the subscript j labels the time, and the superscript α_j labels the different projectors which occur in the decomposition at this time. The decompositions (8.25) for different values of j could be the same or they could be different; they need have no relationship to one another. (Note that the sample spaces for the different classical systems discussed in Sec. 8.2 have this sort of product structure.) Projectors of the form

$$Y^\alpha = P_1^{\alpha_1} \odot P_2^{\alpha_2} \odot \cdots \odot P_f^{\alpha_f}, \tag{8.26}$$

where α is an f-component label

$$\alpha = (\alpha_1, \alpha_2, \ldots \alpha_f), \tag{8.27}$$

make up the sample space, and it is straightforward to check that (8.23) is satisfied.

Here is a simple example for a spin half particle with $f = 2$:

$$I_1 = [z^+] + [z^-], \quad I_2 = [x^+] + [x^-]. \tag{8.28}$$

The product of sample spaces consists of the four histories

$$\begin{aligned} Y^{++} &= [z^+] \odot [x^+], \quad Y^{+-} = [z^+] \odot [x^-], \\ Y^{-+} &= [z^-] \odot [x^+], \quad Y^{--} = [z^-] \odot [x^-], \end{aligned} \tag{8.29}$$

in an obvious notation. The Boolean algebra or family of histories contains $2^4 = 16$ elementary and compound histories, including the null history 0 (which never occurs).

Another type of sample space that arises quite often in practice consists of histories which begin at an initial time t_0 with a specific state represented by a projector Ψ_0, but behave in different ways at later times. We shall refer to it as a *family based upon the initial state* Ψ_0. A relatively simple version is that in which the histories are of the form

$$Y^\alpha = \Psi_0 \odot P_1^{\alpha_1} \odot P_2^{\alpha_2} \odot \cdots \odot P_f^{\alpha_f}, \tag{8.30}$$

with the projectors at times later than t_0 drawn from decompositions of the identity of the type (8.25). The sum over α of the projectors in (8.30) is equal to Ψ_0, so in order to complete the sample space one adds one more history

$$Z = (I - \Psi_0) \odot I \odot I \odot \cdots \odot I \tag{8.31}$$

to the collection. If, as is usually the case, one is only interested in the histories which begin with the initial state Ψ_0, the history Z is assigned zero probability, after which it can be ignored. The procedure for assigning probabilities to the other histories will be discussed in later chapters. Note that histories of the form

$$(I - \Psi_0) \odot P_1^{\alpha_1} \odot P_2^{\alpha_2} \odot \cdots \odot P_f^{\alpha_f} \tag{8.32}$$

are *not* present in the sample space, and for this reason the family of histories based upon an initial state Ψ_0 is distinct from a product of sample spaces in which (8.25) is supplemented with an additional decomposition

$$I_0 = \Psi_0 + (I - \Psi_0) \tag{8.33}$$

at time t_0. As a consequence, later events in a family based upon an initial state Ψ_0 are dependent upon the initial state in the technical sense discussed in Ch. 14.

Other examples of sample spaces which are not products of sample spaces are used in various applications of quantum theory, and some of them will be discussed in later chapters. In all cases the individual histories in the sample space correspond to product projectors on the history space $\breve{\mathcal{H}}$ regarded as a tensor product of Hilbert spaces at different times, (8.5). That is, they are of the form (8.6): a quantum property at t_1, another quantum property at t_2, and so forth. Since the history space $\breve{\mathcal{H}}$ is a Hilbert space, it also contains subspaces which are not of this form, but might be said to be "entangled in time". For example, in the case of a spin-half particle and two times t_1 and t_2, the ket

$$|\epsilon\rangle = \big(|z^+\rangle \odot |z^-\rangle - |z^-\rangle \odot |z^+\rangle\big)/\sqrt{2} \tag{8.34}$$

is an element of $\breve{\mathcal{H}}$, and therefore $[\epsilon] = |\epsilon\rangle\langle\epsilon|$ is a projector on $\breve{\mathcal{H}}$. It seems difficult to find a physical interpretation for histories of this sort, or sample spaces containing such histories.

8.6 Refinements of Histories

The process of refining a sample space in which coarse projectors are replaced with finer projectors on subspaces of lower dimensionality was discussed in Sec. 5.3. Refinement is often used to construct sample spaces of histories, as was noted in connection with the classical random walk in one dimension in Sec. 8.2. Here is a simple example to show how this process works for a quantum system. Consider a spin-half particle and a decomposition of the identity $\{[z^+], [z^-]\}$ at time t_1. Each projector corresponds to a single-time history which can be extended to a second time t_2 in the manner indicated in Sec. 8.3, to make a history sample space containing

$$[z^+] \odot I, \quad [z^-] \odot I. \tag{8.35}$$

If one uses this sample space, there is nothing one can say about the spin of the particle at the second time t_2, since I is always true, and is thus completely uninformative. However, the first projector in (8.35) is the sum of $[z^+] \odot [z^+]$ and $[z^+] \odot [z^-]$, and if one replaces it with these two projectors, and the second projector in (8.35) with the corresponding pair $[z^-]\odot[z^+]$ and $[z^-]\odot[z^-]$, the result is a sample space

$$\begin{aligned} &[z^+] \odot [z^+], \quad [z^+] \odot [z^-], \\ &[z^-] \odot [z^+], \quad [z^-] \odot [z^-], \end{aligned} \tag{8.36}$$

which is a refinement of (8.35), and permits one to say something about the spin at time t_2 as well as at t_1.

When it is possible to refine a sample space in this way, there are always a large number of ways of doing it. Thus the four histories in (8.29) also constitute a refinement of (8.35). However, the refinements (8.29) and (8.36) are mutually incompatible, since it makes no sense to talk about S_x at t_2 at the same time that one is ascribing values to S_z, and vice versa. Both (8.29) and (8.36) are products of sample spaces, but refinements of (8.35) which are not of this type are also possible; for example,

$$\begin{aligned} &[z^+] \odot [z^+], \quad [z^+] \odot [z^-], \\ &[z^-] \odot [x^+], \quad [z^-] \odot [x^-], \end{aligned} \tag{8.37}$$

where the decomposition of the identity used at t_2 is different depending upon which event event occurs at t_1.

The process of refinement can continue by first extending the histories in (8.36) or (8.37) to an additional time, either later than t_2 or earlier than t_1 or between t_1 and t_2, and then replacing the identity I at this additional time with two projectors onto pure states. Note that the process of extension does not by itself lead to a refinement of the sample space, since it leaves the number of histories and their intuitive interpretation unchanged; refinement occurs when I is replaced with projectors on lower-dimensional spaces.

It is important to notice that refinement is *not* some sort of *physical process* which occurs in the quantum system described by these histories. Instead, it is a conceptual process carried out by the quantum physicist in the process of constructing a suitable mathematical description of the time dependence of a quantum system. Unlike deterministic classical mechanics, in which the state of a system at a single time yields a unique description (orbit in the phase space) of what happens at other times, stochastic quantum mechanics allows for a large number of alternative descriptions, and the process of refinement is often a helpful way of selecting useful and interesting sample spaces from among them.

8.7 Unitary Histories

Thus far we have discussed quantum histories without any reference to the dynamical laws of quantum mechanics. The dynamics of histories is not a trivial matter, and is the subject of the next two chapters. However, at this point it is convenient to introduce the notion of a *unitary history*. The simplest example of such a history is the sequence of kets $|\psi_{t_1}\rangle$, $|\psi_{t_2}\rangle$, $\dots |\psi_{t_f}\rangle$, where $|\psi_t\rangle$ is a solution of Schrödinger's equation, Sec. 7.3, or, to be more precise, the corresponding sequence of projectors $[\psi_{t_1}]$, $[\psi_{t_2}]$,.... The general definition is that a history of the form (8.6) is unitary provided

$$F_j = T(t_j, t_1) F_1 T(t_1, t_j) \tag{8.38}$$

is satisfied for $j = 1, 2, \dots f$. That is to say, all the projectors in the history are generated from F_1 by means of the unitary time development operators introduced in Sec. 7.3, see (7.44). In fact, F_1 does not play a distinguished role in this definition and could be replaced by F_k for any k, because for a set of projectors given by (8.38), $T(t_j, t_k) F_k T(t_k, t_j)$ is equal to F_j whatever the value of k.

One can also define *unitary families* of histories. We shall limit ourselves to the case of a product of sample spaces, in the notation of Sec. 8.5, and assume that for each time t_j there is a decomposition of the identity of the form

$$I_j = \sum_a P_j^a. \tag{8.39}$$

The corresponding family is unitary if for each choice of a these projectors satisfy (8.38), that is,

$$P_j^a = T(t_j, t_1) P_1^a T(t_1, t_j) \tag{8.40}$$

for every j. In the simplest (interesting) family of this type each decomposition of the identity contains only two projectors; for example, $[\psi_{t_1}]$ and $I - [\psi_{t_1}]$. Notice that while a unitary family will contain unitary histories, such as

$$P_1^1 \odot P_2^1 \odot P_3^1 \odot \cdots P_f^1, \tag{8.41}$$

it will also contain other histories, such as

$$P_1^1 \odot P_2^2 \odot P_3^1 \odot \ \cdots P_f^1, \tag{8.42}$$

which are not unitary. We will have more to say about unitary histories and families of histories in Secs. 9.3, 9.6, and 10.3.