Chapter 6

Composite Systems and Tensor Products

6.1 Introduction

A *composite* system is one involving more than one particle, or a particle with internal degrees of freedom in addition to its center of mass. In classical mechanics the phase space of a composite system is a *Cartesian product* of the phase spaces of its constituents. The Cartesian product of two sets A and B is the set of (ordered) pairs $\{(a, b)\}$, where a is any element of A and b is any element of B. For three sets A, B, and C the Cartesian product consists of triples $\{(a, b, c)\}$, and so forth. Consider two classical particles in one dimension, with phase spaces x_1, p_1 and x_2, p_2. The phase space for the composite system consists of pairs of points from the two phase spaces, that is, it is a collection of quadruples of the form x_1, p_1, x_2, p_2, which can equally well be written in the order x_1, x_2, p_1, p_2. This is formally the same as the phase space of a single particle in two dimensions, a collection of quadruples x, y, p_x, p_y. Similarly, the 6-dimensional phase space of a particle in 3 dimensions is formally the same as that of three one-dimensional particles.

In quantum theory the analog of a Cartesian product of classical phase spaces is a *tensor product* of Hilbert spaces. A particle in three dimensions has a Hilbert space which is the tensor product of three spaces, each corresponding to motion in one dimension. The Hilbert space for two particles, as long as they are not identical, is the tensor product of the two Hilbert spaces for the separate particles. The Hilbert space for a particle with spin is the tensor product of the Hilbert space of wave functions for the center of mass, appropriate for a particle without spin, with the spin space, which is two-dimensional for a spin-half particle.

Not only are tensor products used in quantum theory for describing a composite system at a single time, they are also very useful for describing the time development of a quantum system, as we shall see in Ch. 8. Hence any serious student of quantum mechanics needs to become familiar with the basic facts about tensor products, and the corresponding notation, which is summarized in Sec. 6.2.

Special rules apply to the tensor product spaces used for identical quantum particles. For identical bosons one uses the symmetrical, while for identical fermions one uses the antisymmetrical subspace of the Hilbert space formed by taking a tensor product of the spaces for the individual particles. The basic procedure for constructing these subspaces is discussed in various introductory and more advanced textbooks (see references in the bibliography), but the idea behind it is probably easiest to understand in the context of quantum field theory, which lies outside the scope of this book. While we shall not discuss the subject further, it is worth pointing out that there are a number of circumstances in which the fact that the particles are identical can be ignored—that is, one makes no significant error by treating them as distinguishable—because they are found in different locations or in different environments. For example, identical nuclei in a solid can be regarded as distinguishable as long as it is a reasonable physical approximation to assume that they are approximately localized, e.g., found in a particular unit cell, or in a particular part of a unit cell. In such cases one can construct the tensor product spaces in a straightforward manner using the principles described below.

6.2 Definition of tensor products

Given two Hilbert spaces $\mathcal{A}$ and $\mathcal{B}$, their tensor product $\mathcal{A}\otimes\mathcal{B}$ can be defined in the following way, where we assume, for simplicity, that the spaces are finite-dimensional. Let $\{|a_j\rangle : j = 1, 2 \dots m\}$ be an orthonormal basis for the m-dimensional space $\mathcal{A}$, and $\{|b_p\rangle : p = 1, 2 \dots n\}$ an orthonormal basis for the n-dimensional space $\mathcal{B}$, so that

$$\langle a_j|a_k\rangle = \delta_{jk}, \quad \langle b_p|b_q\rangle = \delta_{pq}. \tag{6.1}$$

Then the collection of mn elements

$$|a_j\rangle \otimes |b_p\rangle \tag{6.2}$$

forms an orthonormal basis of the tensor product $\mathcal{A}\otimes\mathcal{B}$, which is the set of all linear combinations of the form

$$|\psi\rangle = \sum_j \sum_p \gamma_{jp}\bigl(|a_j\rangle \otimes |b_p\rangle\bigr), \tag{6.3}$$

where the γ_{jp} are complex numbers.

Given kets

$$|a\rangle = \sum_j \alpha_j|a_j\rangle, \quad |b\rangle = \sum_p \beta_p|b_p\rangle \tag{6.4}$$

in $\mathcal{A}$ and $\mathcal{B}$, respectively, their tensor product is defined as

$$|a\rangle \otimes |b\rangle = \sum_j \sum_p \alpha_j\beta_p\bigl(|a_j\rangle \otimes |b_p\rangle\bigr), \tag{6.5}$$

which is of the form (6.3) with

$$\gamma_{jp} = \alpha_j\beta_p. \tag{6.6}$$

The parentheses in (6.3) and (6.5) are not really essential, since $\bigl(\alpha|a\rangle\bigr)\otimes|b\rangle$ is equal to $\alpha\bigl(|a\rangle\otimes|b\rangle\bigr)$, and we shall henceforth omit them when this gives rise to no ambiguities. The definition (6.5) implies that the tensor product operation $\otimes$ is distributive:

$$\begin{aligned} |a\rangle \otimes \bigl(\beta'|b'\rangle + \beta''|b''\rangle\bigr) &= \beta'|a\rangle \otimes |b'\rangle + \beta''|a\rangle \otimes |b''\rangle, \\ \bigl(\alpha'|a'\rangle + \alpha''|a''\rangle\bigr) \otimes |b\rangle &= \alpha'|a'\rangle \otimes |b\rangle + \alpha''|a''\rangle \otimes |b\rangle. \end{aligned} \tag{6.7}$$

An element of $\mathcal{A}\otimes\mathcal{B}$ which can be written in the form $|a\rangle\otimes|b\rangle$ is called a *product* state, and states which are not product states are said to be *entangled.* When several coefficients in (6.3) are non-zero, it may not be readily apparent whether the corresponding state is a product state or entangled, i.e., whether or not γ_{jp} can be written in the form (6.6). For example,

$$1.0|a_1\rangle\otimes|b_1\rangle + 0.5|a_1\rangle\otimes|b_2\rangle - 1.0|a_2\rangle\otimes|b_1\rangle - 0.5|a_2\rangle\otimes|b_2\rangle \tag{6.8}$$

is a product state $\bigl(|a_1\rangle - |a_2\rangle\bigr)\otimes\bigl(|b_1\rangle + 0.5|b_2\rangle\bigr)$, whereas changing the sign of the last coefficient yields an entangled state

$$1.0|a_1\rangle\otimes|b_1\rangle + 0.5|a_1\rangle\otimes|b_2\rangle - 1.0|a_2\rangle\otimes|b_1\rangle + 0.5|a_2\rangle\otimes|b_2\rangle. \tag{6.9}$$

The linear functional or bra vector corresponding to the product state $|a\rangle\otimes|b\rangle$ is written as

$$\bigl(|a\rangle\otimes|b\rangle\bigr)^\dagger = \langle a|\otimes\langle b|, \tag{6.10}$$

where the $\mathcal{A}\otimes\mathcal{B}$ order of the factors on either side of $\otimes$ does not change when the dagger operation is applied. The result for a general linear combination (6.3) follows from (6.10) and the antilinearity of the dagger operation:

$$\langle\psi| = \bigl(|\psi\rangle\bigr)^\dagger = \sum_{jp}\gamma_{jp}^*\langle a_j|\otimes\langle b_p|. \tag{6.11}$$

Consistent with these formulas, the inner product of two product states is given by

$$\bigl(|a\rangle\otimes|b\rangle\bigr)^\dagger\bigl(|a'\rangle\otimes|b'\rangle\bigr) = \langle a|a'\rangle\cdot\langle b|b'\rangle, \tag{6.12}$$

and of a general state $|\psi\rangle$, (6.3), with another state

$$|\psi'\rangle = \sum_{jp}\gamma'_{jp}|a_j\rangle\otimes|b_p\rangle, \tag{6.13}$$

by the expression

$$\langle\psi|\psi'\rangle = \sum_{jp}\gamma_{jp}^*\gamma'_{jp}. \tag{6.14}$$

Because the definition of a tensor product given above employs specific orthonormal bases for $\mathcal{A}$ and $\mathcal{B}$, one might suppose that the space $\mathcal{A}\otimes\mathcal{B}$ somehow depends on the choice of these bases. But in fact it does not, as can be seen by considering alternative bases $\{|a'_k\rangle\}$ and $\{|b'_q\rangle\}$. The kets in the new bases can be written as linear combinations of the original kets,

$$|a'_k\rangle = \sum_j\langle a_j|a'_k\rangle\cdot|a_j\rangle,\quad |b'_q\rangle = \sum_p\langle b_p|b'_q\rangle\cdot|b_p\rangle, \tag{6.15}$$

and (6.5) then allows $|a'_k\rangle\otimes|b'_q\rangle$ to be written as a linear combination of the kets $|a_j\rangle\otimes|b_p\rangle$. Hence the use of different bases for $\mathcal{A}$ or $\mathcal{B}$ leads to the same tensor product space $\mathcal{A}\otimes\mathcal{B}$, and it is easily checked that the property of being a product state or an entangled state does not depend upon the choice of bases.

Just as for any other Hilbert space, it is possible to choose an orthonormal basis of $\mathcal{A}\otimes\mathcal{B}$ in a large number of different ways. We shall refer to a basis of the type used in the original definition, $\{|a_j\rangle\otimes|b_p\rangle\}$, as a *product of bases*. An orthonormal basis of $\mathcal{A}\otimes\mathcal{B}$ may consist entirely of product states without being a product of bases; see the example in (6.22). Or it might consist entirely of entangled states, or of some entangled states and some product states.

Physicists often omit the $\otimes$ and write $|a\rangle\otimes|b\rangle$ in the form $|a\rangle|b\rangle$, or more compactly as $|a,b\rangle$, or even as $|ab\rangle$. Any of these notations is perfectly adequate when it is clear from the context that a tensor product is involved. We shall often use one of the more compact notations, and occasionally insert the $\otimes$ symbol for the sake of clarity, or for emphasis. Note that while a double label inside a ket, as in $|a,b\rangle$, often indicates a tensor product, this is not always the case; for example, the double label $|l,m\rangle$ for orbital angular momentum kets does not signify a tensor product.

The tensor product of three or more Hilbert spaces can be obtained by an obvious generalization of the ideas given above. In particular, the tensor product $\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{C}$ of three Hilbert spaces $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, consists of all linear combinations of states of the form

$$|a_j\rangle\otimes|b_p\rangle\otimes|c_s\rangle, \tag{6.16}$$

using the bases introduced earlier, together with $\{|c_s\rangle : s = 1,2\ldots\}$, an orthonormal basis for $\mathcal{C}$. One can think of $\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{C}$ as obtained by first forming the tensor product of two of the spaces, and then taking the tensor product of this space with the third. The final result does not depend upon which spaces form the initial pairing:

$$\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{C} = (\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C} = \mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C}) = (\mathcal{A}\otimes\mathcal{C})\otimes\mathcal{B}. \tag{6.17}$$

In what follows we shall usually focus on tensor products of two spaces, but for the most part the discussion can be generalized in an obvious way to tensor products of three or more spaces. Where this is not the case it will be pointed out explicitly.

Given *any* state $|\psi\rangle$ in $\mathcal{A}\otimes\mathcal{B}$, it is always possible to find particular orthonormal bases $\{|\hat{a}_j\rangle\}$ for $\mathcal{A}$ and $\{|\hat{b}_p\rangle\}$ for $\mathcal{B}$ such that $|\psi\rangle$ takes the form

$$|\psi\rangle = \sum_j \lambda_j |\hat{a}_j\rangle\otimes|\hat{b}_j\rangle. \tag{6.18}$$

Here the λ_j are complex numbers, but by choosing appropriate phases for the basis states, one can make them real and non-negative. The summation index j takes values between 1 and the minimum of the dimensions of $\mathcal{A}$ and $\mathcal{B}$. The result (6.18) is known as the *Schmidt decomposition* of $|\psi\rangle$; it is also referred to as the *biorthogonal* or *polar expansion* of $|\psi\rangle$. It does *not* generalize, at least in any simple way, to a tensor product of three or more Hilbert spaces.

Given an arbitrary Hilbert space $\mathcal{H}$ of dimension mn, with m and n integers greater than one, it is possible to "decompose" it into a tensor product $\mathcal{A}\otimes\mathcal{B}$, with m the dimension of $\mathcal{A}$ and n the dimension of $\mathcal{B}$; indeed, this can be done in many different ways. Let $\{|h_l\rangle\}$ be any orthonormal basis of $\mathcal{H}$, with $l = 1,2\ldots mn$. Rather than use a single label for the kets, we can associate each l with a pair j,p, where j takes values between 1 and m, and p values between 1 and n. Any association will do, as long as it is unambiguous (one to one). Let $\{|h_{jp}\rangle\}$ denote precisely the same basis using this new labeling. Now write

$$|h_{jp}\rangle = |a_j\rangle\otimes|b_p\rangle, \tag{6.19}$$

where the $\{|a_j\rangle\}$ for j between 1 and m are defined to be an orthonormal basis of a Hilbert space $\mathcal{A}$, and the $\{|b_p\rangle\}$ for p between 1 and n the orthonormal basis of a Hilbert space $\mathcal{B}$. By this process we have turned $\mathcal{H}$ into a tensor product $\mathcal{A}\otimes\mathcal{B}$, or it might be better to say that we have imposed a tensor product structure $\mathcal{A}\otimes\mathcal{B}$ upon the Hilbert space $\mathcal{H}$. In the same way, if the dimension of $\mathcal{H}$ is the product of three or more integers greater than 1, it can always be thought of as a tensor product of three or more spaces, and the decomposition can be carried out in many different ways.

6.3 Examples of Composite Quantum Systems

Figure 6.1(a) shows a toy model involving two particles. The first particle can be at any one of the $M = 6$ sites indicated by circles, and the second particle can be at one of the two sites indicated by squares. The states $|m\rangle$ for m between 0 and 5 span the Hilbert space $\mathcal{M}$ for the first particle, and $|n\rangle$ for $n = 0, 1$ the Hilbert space $\mathcal{N}$ for the second particle. The tensor product space $\mathcal{M}\otimes\mathcal{N}$ is $6 \times 2 = 12$ dimensional, with basis states $|m\rangle \otimes |n\rangle = |m, n\rangle$. (In Sec. 7.4 we shall put this arrangement to good use: the second particle will be employed as a detector to detect the passage of the first particle.) One must carefully distinguish the case of *two* particles, one located on the circles and one on the squares in Fig. 6.1(a), from that of a *single* particle which can be located on either the circles or the squares. The former has a Hilbert space of dimension 12, and the latter a Hilbert space of dimension $6 + 2 = 8$.

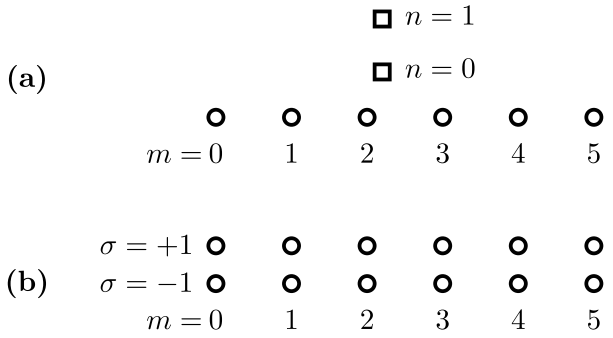

Figure 6.1: Toy model for (a) two particles, one located on the circles and one on the squares; (b) a particle with an internal degree of freedom.

A second toy model, Fig. 6.1(b), consists of a *single* particle with an internal degree of freedom represented by a "spin" variable which can take on two possible values. The center of mass of the particle can be at any one of 6 sites corresponding to a 6-dimensional Hilbert space $\mathcal{M}$, whereas the the spin degree of freedom is represented by a 2-dimensional Hilbert space $\mathcal{S}$. The basis kets of $\mathcal{M}\otimes\mathcal{S}$ have the form $|m, \sigma\rangle$, with $\sigma = \pm 1$. The figure shows two circles at each site, one corresponding to $\sigma = +1$ ("spin up"), and the other to $\sigma = -1$ ("spin down"), so one can think of each basis state as the particle being "at" one of the circles. A general element $|\psi\rangle$ of the Hilbert space $\mathcal{M}\otimes\mathcal{S}$ is a linear combination of the basis kets, so it can be written in the form

$$|\psi\rangle = \sum_m \sum_\sigma \psi(m, \sigma)|m, \sigma\rangle, \tag{6.20}$$

where the complex coefficients $\psi(m,\sigma)$ form a toy wave function; this is simply an alternative way of writing the complex coefficients γ_{jp} in (6.3). The toy wave function $\psi(m,\sigma)$ can be thought of as a discrete analog of the wave function $\psi(\boldsymbol{r},\sigma)$ used to describe a spin-half particle in three dimensions. Just as in the toy model, the Hilbert space to which $\psi(\boldsymbol{r},\sigma)$ belongs is a tensor product of the space of wave functions $\psi(\boldsymbol{r})$, appropriate for a spinless quantum particle particle, with the two-dimensional spin space.

Consider two spin-half particles a and b, such as an electron and a proton, and ignore their center-of-mass degrees of freedom. The tensor product $\mathcal{H}$ of the two 2-dimensional spin spaces is a four dimensional space spanned by the orthonormal basis

$$|z_a^+\rangle \otimes |z_b^+\rangle, \quad |z_a^+\rangle \otimes |z_b^-\rangle, \quad |z_a^-\rangle \otimes |z_b^+\rangle, \quad |z_a^-\rangle \otimes |z_b^-\rangle \tag{6.21}$$

in the notation of Sec. 4.2. This is a product of bases in the terminology of Sec. 6.2. By contrast, the basis

$$|z_a^+\rangle \otimes |z_b^+\rangle, \quad |z_a^+\rangle \otimes |z_b^-\rangle, \quad |z_a^-\rangle \otimes |x_b^+\rangle, \quad |z_a^-\rangle \otimes |x_b^-\rangle, \tag{6.22}$$

whereas it consists of product states, is *not* a product of bases, because one basis for $\mathcal{B}$ is employed along with $|z_a^+\rangle$, and a different basis along with $|z_a^-\rangle$. Still other bases are possible, including cases in which some or all of the basis vectors are entangled states.

The spin space for three spin-half particles a, b, and c is an 8-dimensional tensor product space, and the state $|z_a^+\rangle \otimes |z_b^+\rangle \otimes |z_c^+\rangle$ along with the seven other states in which some of the pluses are replaced by minuses forms a product basis. For N spins, the tensor product space is of dimension 2^N.

6.4 Product Operators

Since $\mathcal{A} \otimes \mathcal{B}$ is a Hilbert space, operators on it obey all the usual rules, Sec. 3.3. What we are interested in is how these operators are related to the tensor product structure, and, in particular, to operators on the separate factor spaces $\mathcal{A}$ and $\mathcal{B}$. In this section we discuss the special case of product operators, while general operators are considered in the next section. The considerations which follow can be generalized in an obvious way to a tensor product of three or more spaces.

If A is an operator on $\mathcal{A}$ and B an operator on $\mathcal{B}$, the *(tensor) product operator* $A \otimes B$ acting on a product state $|a\rangle \otimes |b\rangle$ yields another product state:

$$(A \otimes B)\big(|a\rangle \otimes |b\rangle\big) = \big(A|a\rangle\big) \otimes \big(B|b\rangle\big). \tag{6.23}$$

Since $A \otimes B$ is by definition a linear operator on $\mathcal{A} \otimes \mathcal{B}$, one can use (6.23) to define its action on a general element $|\psi\rangle$, (6.3), of $A \otimes B$:

$$(A \otimes B)\Big[\sum_{jp} \gamma_{jp}\big(|a_j\rangle \otimes |b_p\rangle\big)\Big] = \sum_{jp} \gamma_{jp}\big(A|a_j\rangle \otimes B|b_p\rangle\big). \tag{6.24}$$

The tensor product of two operators which are themselves sums of other operators can be written as a sum of product operators using the usual distributive rules. Thus:

$$\begin{aligned} (\alpha A + \alpha' A') \otimes B &= \alpha(A \otimes B) + \alpha'(A' \otimes B), \\ A \otimes (\beta B + \beta' B') &= \beta(A \otimes B) + \beta'(A \otimes B'). \end{aligned} \tag{6.25}$$

The parentheses on the right side are not essential, as there is no ambiguity when $\alpha(A \otimes B) = (\alpha A) \otimes B$ is written as $\alpha A \otimes B$.

If $|\psi\rangle = |a\rangle \otimes |b\rangle$ and $|\phi\rangle = |a'\rangle \otimes |b'\rangle$ are both product states, the dyad $|\psi\rangle\langle\phi|$ is a product operator:

$$\big(|a\rangle \otimes |b\rangle\big)\big(\langle a'| \otimes \langle b'|\big) = \big(|a\rangle\langle a'|\big) \otimes \big(|b\rangle\langle b'|\big). \tag{6.26}$$

Notice how the terms on the left are rearranged in order to arrive at the expression on the right. One can omit the parentheses on the right side, since $|a\rangle\langle a'| \otimes |b\rangle\langle b'|$ is unambiguous.

The adjoint of a product operator is the tensor product of the adjoints *in the same order* relative to the symbol $\otimes$:

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger. \tag{6.27}$$

Of course, if the operators on $\mathcal{A}$ and $\mathcal{B}$ are themselves products, one must reverse their order when taking the adjoint:

$$(A_1A_2 \otimes B_1B_2B_3)^\dagger = A_2^\dagger A_1^\dagger \otimes B_3^\dagger B_2^\dagger B_1^\dagger. \tag{6.28}$$

The ordinary operator product of two tensor product operators is given by

$$(A \otimes B) \cdot (A' \otimes B') = AA' \otimes BB', \tag{6.29}$$

where it is important that the order of the operators be preserved: A is to the left of A' on both sides of the equation, and likewise B is to the left of B'. An operator product of sums of tensor products can be worked out using the usual distributive law, e.g.,

$$(A \otimes B) \cdot (A' \otimes B' + A'' \otimes B'') = AA' \otimes BB' + AA'' \otimes BB''. \tag{6.30}$$

An operator A on $\mathcal{A}$ can be *extended* to an operator $A \otimes I_{\mathcal{B}}$ on $\mathcal{A}\otimes\mathcal{B}$, where $I_{\mathcal{B}}$ is the identity on $\mathcal{B}$. It is customary to use the same symbol, A, for both the original operator and its extension; indeed, in practice it would often be quite awkward to do anything else. Similarly, B is used to denote either an operator on $\mathcal{B}$, or its extension $I_{\mathcal{A}} \otimes B$. Consider, for example, two spin-half particles, an electron and a proton. It is convenient to use the symbol S_{ez} for the operator corresponding to the z component of the spin of the electron, whether one is thinking of the two-dimensional Hilbert space associated with the electron spin by itself, the four-dimensional spin space for both particles, the infinite-dimensional space of electron space-and-spin wave functions, or the space needed to describe the spin and position of both the electron and the proton.

Using the same symbol for an operator and its extension normally causes no confusion, since the space to which the operator is applied will be evident from the context. However, it is sometimes useful to employ the longer notation for clarity or emphasis, in which case one can (usually) omit the subscript from the identity operator: in the operator $A \otimes I$ it is clear that I is the identity on $\mathcal{B}$. Note that (6.29) allows one to write

$$A \otimes B = (A \otimes I) \cdot (I \otimes B) = (I \otimes B) \cdot (A \otimes I), \tag{6.31}$$

and hence if we use A for $A \otimes I$ and B for $I \otimes B$, $A \otimes B$ can be written as the operator product AB or BA. This is perfectly correct and unambiguous as long as it is clear from the context that A is an operator on $\mathcal{A}$ and B an operator on $\mathcal{B}$. However, if $\mathcal{A}$ and $\mathcal{B}$ are identical (isomorphic) spaces, and B denotes an operator which also makes sense on $\mathcal{A}$, then AB could be interpreted as the ordinary product of two operators on $\mathcal{A}$ (or on $\mathcal{B}$), and to avoid confusion it is best to use the unabbreviated $A \otimes B$.

6.5 General Operators, Matrix Elements, Partial Traces

Any operator on a Hilbert space is uniquely specified by its matrix elements in some orthonormal basis, Sec. 3.6, and thus a general operator D on $\mathcal{A} \otimes \mathcal{B}$ is determined by its matrix elements in the orthonormal basis (6.2). These can be written in a variety of different ways:

$$\begin{aligned} \langle jp|D|kq\rangle &= \langle j,p|D|k,q\rangle = \langle a_j b_p|D|a_k b_q\rangle \\ &= \big(\langle a_j| \otimes \langle b_p|\big) D \big(|a_k\rangle \otimes |b_q\rangle\big). \end{aligned} \tag{6.32}$$

The most compact notation is on the left, but it is not always the clearest. Note that it corresponds to writing bras and kets with a "double label", and this needs to be taken into account in standard formulas, such as

$$I = I \otimes I = \sum_j \sum_p |jp\rangle\langle jp| \tag{6.33}$$

and

$$\mathrm{Tr}(D) = \sum_j \sum_p \langle jp|D|jp\rangle, \tag{6.34}$$

which correspond to (3.54) and (3.79), respectively.

Any operator can be written as a sum of dyads multiplied by appropriate matrix elements, (3.67), which allows us to write

$$D = \sum_{jk} \sum_{pq} \langle a_j b_p|D|a_k b_q\rangle \big(|a_j\rangle\langle a_k| \otimes |b_p\rangle\langle b_q|\big), \tag{6.35}$$

where we have used (6.26) to rewrite the dyads as product operators. This shows, incidentally, that while not all operators on $\mathcal{A} \otimes \mathcal{B}$ are product operators, any operator can be written as a sum of product operators. The adjoint of D is then given by the formula

$$D^\dagger = \sum_{jk} \sum_{pq} \langle a_j b_p|D|a_k b_q\rangle^* \big(|a_k\rangle\langle a_j| \otimes |b_q\rangle\langle b_p|\big), \tag{6.36}$$

using (6.27) and the fact the dagger operation is antilinear. If one replaces $\langle a_j b_p|D|a_k b_q\rangle^*$ by $\langle a_k b_q|D^\dagger|a_j b_p\rangle$, see (3.64), (6.36) is simply (6.35) with D replaced by $D^\dagger$ on both sides, aside from dummy summation indices.

The matrix elements of a product operator using the basis (6.2) are the products of the matrix elements of the factors:

$$\langle a_j b_p|A \otimes B|a_k b_q\rangle = \langle a_j|A|a_k\rangle \cdot \langle b_p|B|b_q\rangle. \tag{6.37}$$

From this it follows that the trace of a product operator is the product of the traces of its factors:

$$\mathrm{Tr}[A \otimes B] = \sum_j \langle a_j|A|a_j\rangle \cdot \sum_p \langle b_p|B|b_p\rangle = \mathrm{Tr}_{\mathcal{A}}[A] \cdot \mathrm{Tr}_{\mathcal{B}}[B]. \tag{6.38}$$

Here the subscripts on $\mathrm{Tr}_{\mathcal{A}}$ and $\mathrm{Tr}_{\mathcal{B}}$ indicate traces over the spaces $\mathcal{A}$ and $\mathcal{B}$, respectively, while the trace over $\mathcal{A} \otimes \mathcal{B}$ is written without a subscript, though one could denote it by $\mathrm{Tr}_{\mathcal{AB}}$ or $\mathrm{Tr}_{\mathcal{A}\otimes\mathcal{B}}$. Thus if $\mathcal{A}$ and $\mathcal{B}$ are spaces of dimension m and n, $\mathrm{Tr}_{\mathcal{A}}[I] = m$, $\mathrm{Tr}_{\mathcal{B}}[I] = n$, and $\mathrm{Tr}[I] = mn$.

Given an operator D on $\mathcal{A}\otimes\mathcal{B}$, and two basis states $|b_p\rangle$ and $|b_q\rangle$ of $\mathcal{B}$, one can define $\langle b_p|D|b_q\rangle$ to be the (unique) operator on $\mathcal{A}$ which has matrix elements

$$\langle a_j|\Big(\langle b_p|D|b_q\rangle\Big)|a_k\rangle = \langle a_j b_p|D|a_k b_q\rangle. \tag{6.39}$$

The *partial trace* over $\mathcal{B}$ of the operator D is defined to be a sum of operators of this type:

$$D_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{B}}[D] = \sum_p \langle b_p|D|b_p\rangle. \tag{6.40}$$

Alternatively, one can define $D_{\mathcal{A}}$ to be the operator on $\mathcal{A}$ with matrix elements

$$\langle a_j|D_{\mathcal{A}}|a_k\rangle = \sum_p \langle a_j b_p|D|a_k b_p\rangle. \tag{6.41}$$

Note that the $\mathcal{B}$ state labels are the same on both sides of the matrix elements on the right hand sides of (6.40) and (6.41), while those for the $\mathcal{A}$ states are (in general) different. Even though we have employed a specific orthonormal basis of $\mathcal{B}$ in (6.40) and (6.41), it is not hard to show that the partial trace $D_{\mathcal{A}}$ is independent of this basis; that is, one obtains precisely the same operator if a different orthonormal basis $\{|b'_p\rangle\}$ is used in place of $\{|b_p\rangle\}$.

If D is written in the form (6.35), its partial trace is

$$D_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{B}}[D] = \sum_{jk} d_{jk}|a_j\rangle\langle a_k|, \tag{6.42}$$

where

$$d_{jk} = \sum_p \langle a_j b_p|D|a_k b_p\rangle, \tag{6.43}$$

since the trace over $\mathcal{B}$ of $|b_p\rangle\langle b_q|$ is $\langle b_p|b_q\rangle = \delta_{pq}$. In the special case of a product operator $A\otimes B$, the partial trace over $\mathcal{B}$ yields an operator

$$\mathrm{Tr}_{\mathcal{B}}[A\otimes B] = (\mathrm{Tr}_{\mathcal{B}}[B])\, A \tag{6.44}$$

proportional to A.

In a similar way, the partial trace of an operator D on $\mathcal{A}\otimes\mathcal{B}$ over $\mathcal{A}$ yields an operator

$$D_{\mathcal{B}} = \mathrm{Tr}_{\mathcal{A}}[D] \tag{6.45}$$

acting on the space $\mathcal{B}$, with matrix elements

$$\langle b_p|D_{\mathcal{B}}|b_q\rangle = \sum_j \langle a_j b_p|D|a_j b_q\rangle. \tag{6.46}$$

Note that the full trace of D over $\mathcal{A}\otimes\mathcal{B}$ can be written as a trace of either of its partial traces:

$$\mathrm{Tr}[D] = \mathrm{Tr}_{\mathcal{A}}[D_{\mathcal{A}}] = \mathrm{Tr}_{\mathcal{B}}[D_{\mathcal{B}}]. \tag{6.47}$$

All of the above can be generalized to a tensor product of three or more spaces in an obvious way. For example, if E is an operator on $\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{C}$, its matrix elements using the orthonormal product of bases in (6.16) are of the form

$$\langle jpr|E|kqs\rangle = \langle a_j b_p c_r|E|a_k b_q c_s\rangle. \tag{6.48}$$

The partial trace of E over $\mathcal{C}$ is an operator on $\mathcal{A}\otimes\mathcal{B}$, while its partial trace over $\mathcal{B}\otimes\mathcal{C}$ is an operator on $\mathcal{A}$, etc.

6.6 Product Properties and Product of Sample Spaces

Let A and B be projectors representing properties of two physical systems $\mathcal{A}$ and $\mathcal{B}$, respectively. It is easy to show that

$$P = A \otimes B = (A \otimes I) \cdot (I \otimes B) \tag{6.49}$$

is a projector, which therefore represents some property on the tensor product space $\mathcal{A} \otimes \mathcal{B}$ of the combined system. (Note that if A projects onto a pure state $|a\rangle$ and B onto a pure state $|b\rangle$, then P projects onto the pure state $|a\rangle \otimes |b\rangle$.) The physical significance of P is that $\mathcal{A}$ has the property A and $\mathcal{B}$ has the property B. In particular, the projector $A \otimes I$ has the significance that $\mathcal{A}$ has the property A without reference to the system $\mathcal{B}$, since the identity $I = I_{\mathcal{B}}$ operator in $A \otimes I$ is the property which is always true for $\mathcal{B}$, and thus tells us nothing whatsoever about $\mathcal{B}$. Similarly, $I \otimes B$ means that $\mathcal{B}$ has the property B without reference to the system $\mathcal{A}$. The product of $A \otimes I$ with $I \otimes B$—note that the two operators commute with each other—represents the conjunction of the properties of the separate subsystems, in agreement with the discussion in Sec. 4.5, and consistent with the interpretation of P given previously. As an example, consider two spin-half particles a and b. The projector $[z_a^+] \otimes [x_b^-]$ means that $S_{az} = +1/2$ for particle a and $S_{bx} = -1/2$ for particle b.

The interpretation of projectors on $\mathcal{A} \otimes \mathcal{B}$ which are *not* products of projectors is more subtle. Consider, for example, the entangled state

$$|\psi\rangle = \bigl(|z_a^+\rangle|z_b^-\rangle - |z_a^-\rangle|z_b^+\rangle\bigr)/\sqrt{2} \tag{6.50}$$

of two spin-half particles, and let $[\psi]$ be the corresponding dyad projector. Since $[\psi]$ projects onto a subspace of $\mathcal{A} \otimes \mathcal{B}$, it represents some property of the combined system. However, if we ask what this property means in terms of the a spin by itself, we run into the difficulty that the only projectors on the two-dimensional spin space $\mathcal{A}$ which commute with $[\psi]$ are 0 and the identity I. Consequently, any "interesting" property of $\mathcal{A}$, something of the form $S_{aw} = +1/2$ for some direction w, is incompatible with $[\psi]$. Thus $[\psi]$ cannot be interpreted as meaning that the a spin has some property, and likewise it cannot mean that the b spin has some property.

The same conclusion applies to *any* entangled state of two spin-half particles. The situation is not quite as bad if one goes to higher-dimensional spaces. For example, the projector $[\phi]$ corresponding to the entangled state

$$|\phi\rangle = \bigl(|1\rangle \otimes |0\rangle + |2\rangle \otimes |1\rangle\bigr)/\sqrt{2} \tag{6.51}$$

of the toy model with two particles shown in Fig. 6.1(a) commutes with the projector

$$\bigl([1] + [2]\bigr) \otimes I \tag{6.52}$$

for the first particle, and thus if the combined system is described by $[\phi]$, one can say that the first particle is not outside the interval containing the sites $m = 1$ and $m = 2$, although it cannot be assigned a location at one or the other of these sites. However, one can say nothing interesting about the second particle.

A *product of sample spaces* or *product of decompositions* is a collection of projectors $\{A_j \otimes B_p\}$ which sum to the identity

$$I = \sum_{jp} A_j \otimes B_p \tag{6.53}$$

of $\mathcal{A} \otimes \mathcal{B}$, where $\{A_j\}$ is decomposition of the identity for $\mathcal{A}$, and $\{B_p\}$ a decomposition of the identity for $\mathcal{B}$. Note that the event algebra corresponding to (6.53) contains all projectors of the form $\{A_j \otimes I\}$ or $\{I \otimes B_p\}$, so these properties of the individual systems make sense in a description of the composite system based upon this decomposition. A particular example of a product of sample spaces is the collection of dyads corresponding to the product of bases in (6.2):

$$I = \sum_{jp} |a_j\rangle\langle a_j| \otimes |b_p\rangle\langle b_p|. \tag{6.54}$$

A decomposition of the identity can consist of products of projectors without being a product of sample spaces. An example is provided by the four projectors

$$[z_a^+] \otimes [z_b^+], \quad [z_a^+] \otimes [z_b^-], \quad [z_a^-] \otimes [x_b^+], \quad [z_a^-] \otimes [x_b^-] \tag{6.55}$$

corresponding to the states in the basis (6.22) for two spin-half particles. (As noted earlier, (6.22) is not a product of bases.) The event algebra generated by (6.55) contains the projectors $[z_a^+] \otimes I$ and $[z_a^-] \otimes I$, but it does not contain the projectors $I \otimes [z_b^+]$, $I \otimes [z_b^-]$, $I \otimes [x_b^+]$ or $I \otimes [x_b^-]$. Consequently one has the odd situation that if the state $[z_a^-] \otimes [x_b^+]$, which would normally be interpreted to mean $S_{az} = -1/2$ AND $S_{bx} = +1/2$, is a correct description of the system, then using the event algebra based upon (6.55), one can infer that $S_{az} = -1/2$ for spin a, but one cannot infer that $S_{bx} = +1/2$ is a property of spin b by itself, independent of any reference to spin a. Further discussion of this peculiar state of affairs, which arises when one is dealing with *dependent* or *contextual* properties, will be found in Ch. 14.