Harmonic Oscillator

Robert B. Griffiths
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1 Notation

⋆ It is convenient to introduce dimensionless quantities when discussing the quantum harmonic oscillator. Here is the notation which will be used in these notes.

- Hamiltonian in terms of dimensioned quantities:
  \[
  H = \frac{1}{2m} P^2 + \frac{1}{2} \hbar X^2 = \frac{1}{2m} P^2 + \frac{1}{2} m\omega^2 X^2 = \frac{1}{2} \left( -\frac{\hbar^2}{m} \frac{\partial^2}{\partial x^2} + m\omega^2 X^2 \right)
  \]  

- The quantum oscillator has a characteristic length:
  \[
  \xi = \sqrt{\frac{\hbar}{m\omega}}
  \]  

- Characteristic momentum:
  \[
  \eta = \sqrt{\hbar m\omega}
  \]  

- Characteristic velocity:
  \[
  \nu = \sqrt{\hbar\omega/m}
  \]  

○ Note dimensions: $\hbar \sim ml^2 t^{-1}$ implies that $\xi \sim l$, $\eta \sim ml t^{-1}$, $\nu \sim lt^{-1}$.

○ Note that $\xi \eta = \hbar$, $\nu = \eta/m$.

- Placing a bar over the symbol for an operator gives its dimensionless counterpart:
  \[
  \bar{X} = X/\xi, \quad \bar{P} = P/\eta, \quad \bar{H} = H/\hbar\omega = \frac{1}{2} (\bar{P}^2 + \bar{X}^2)
  \]  

○ The commutator $[X,P] = i\hbar I$ becomes $[\bar{X},\bar{P}] = i I$, with $I$ the identity operator.

- Lowering (destruction) $a$ and raising (creation) $a^\dagger$ operators are defined as:
  \[
  a = \frac{1}{\sqrt{2}} (\bar{X} + i \bar{P}), \quad a^\dagger = \frac{1}{\sqrt{2}} (\bar{X} - i \bar{P})
  \]  

○ Note that since $\bar{X}$ and $\bar{P}$ are Hermitian, the definition of $a^\dagger$ follows from that of $a$, and vice versa.
In terms of \( a^\dagger \) and \( a \):
\[
\begin{align*}
\bar{X} &= \frac{1}{\sqrt{2}}(a + a^\dagger), \quad \bar{P} = \frac{1}{i\sqrt{2}}(a - a^\dagger), \\
N &= a^\dagger a, \quad \bar{H} = \frac{1}{2}(a^\dagger a + a a^\dagger) = a^\dagger a + \frac{1}{2} = N + \frac{1}{2}.
\end{align*}
\]

**Dimensionless position representation:**
\[
\begin{align*}
\bar{X} &= \text{multiplication by } u, \quad \bar{P} = -i\partial/\partial u, \\
a &= \frac{1}{\sqrt{2}}(u + \partial/\partial u), \quad a^\dagger = \frac{1}{\sqrt{2}}(u - \partial/\partial u), \quad \bar{H} = \frac{1}{2}\left(-\frac{\partial^2}{\partial u^2} + u^2\right).
\end{align*}
\]

**Dimensionless momentum representation:**
\[
\begin{align*}
\bar{X} &= \text{multiplication by } u, \quad \bar{P} = -i\partial/\partial v, \\
\bar{H} &= \frac{1}{2}\left(\partial^2 - \frac{\partial}{\partial v^2}\right).
\end{align*}
\]

## 2 Eigenstates of the Number Operator \( N \)

\* See the treatment in Townsend Sec. 7.3, which is also found in many other textbooks. The conclusion is that the eigenvalues of \( N \) must be nonnegative integers. Given some eigenket of \( N \), others can be generated using raising and lowering operators. In particular there is a minimum eigenvalue of 0, and if \(|0\rangle\) denotes the corresponding ket, one can generate an infinite collection of orthogonal kets:
\[
|n\rangle = (1/\sqrt{n!})(a^\dagger)^n|0\rangle; \quad N|n\rangle = n|n\rangle; \quad \bar{H}|n\rangle = (n + \frac{1}{2})|n\rangle.
\]

The factor of \( 1/\sqrt{n!} \) is needed for normalization, and the +1 phase in this definition of \(|n\rangle\) is the standard convention.

## 3 Position and Momentum Representations of Number Eigenstates

\* See the treatment in Townsend Sec. 7.4. The ground state in the dimensionless position representation, \( \phi_0(u) = \langle u|0\rangle \), is the solution to a differential equation, and when normalized (and using the conventional phase) it takes the form
\[
\phi_0(u) = (\pi)^{-1/4}\exp[-u^2/2].
\]

**Starting with \( \phi_0(u) \) the other number eigenstates can be obtained by applying \( a^\dagger \) an appropriate number of times, see (13), where \( a^\dagger \) is the differential operator given in (10). The result is that
\[
\langle u|n\rangle = \phi_n(u) \propto h_n(u)\exp[-u^2/2],
\]
where \( h_n(u) \) is the Hermite polynomial of degree \( n \).

**Number eigenstates in the dimensionless momentum representation can be obtained from those in the position representation, and vice versa, by means of a Fourier transform:**
\[
\begin{align*}
\langle v|n\rangle &= \hat{\phi}_n(v) = \int_{-\infty}^{\infty} \langle v|u\rangle \phi_n(u) \, du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iuv} \phi_n(u) \, du; \\
\langle u|n\rangle &= \phi_n(u) = \int_{-\infty}^{\infty} \langle u|v\rangle \phi_n(v) \, dv = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iuv} \hat{\phi}_n(v) \, dv.
\end{align*}
\]

**It turns out that the wave functions for dimensionless momentum and position corresponding to \(|n\rangle\) are identical apart from a phase:
\[
\hat{\phi}_n(v) = (-i)^n \phi_n(u = v).
\]
That is, start with the position space function \( \phi_n(u) \), replace \( u \) everywhere with \( v \), and multiply by \((-i)^n\) to obtain \( \phi_n(v) \).

- In particular, for \( n = 0 \),
  \[
  \hat{\phi}_0(v) = (\pi)^{-1/4} \exp[-v^2/2] 
  \]  
  (19)

\(\Box\) Exercise. Assuming (16), prove the connection between \( \hat{\phi}_n(v) \) and \( \phi_n(u) \) stated in (18). [Hint: For \( n = 0 \) carry out the integral in (16). Then note that \( \langle v|n \rangle \) can be obtained from \( \langle v|0 \rangle \) by repeated applications of \( a^\dagger \) in the momentum representation, see (12).]

## 4 Coherent States

### 4.1 Definition, properties, time dependence

\(\star\) Quantum states of a harmonic oscillator that actually oscillate in time cannot be energy eigenstates, which are stationary. The **coherent states** of a harmonic oscillator exhibit a temporal behavior which is similar to what one observes in a classical oscillator.

\(\star\) Definition. Let \( \alpha \) by any complex number. Define

\[
|\alpha\rangle := e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle
\]

(20)

- Coherent state kets \( |\alpha\rangle \) should not be confused with number states \( |n\rangle \). The nature of the argument (complex number vs. integer) can serve to distinguish them, or one can add an identifying subscript \( |\alpha\rangle_c \) in cases where there might be confusion. In fact the states \( |\alpha = 0\rangle \) and \( |n = 0\rangle \) are identical, so at least they do not need to be distinguished.

\(\Box\) Exercise. Show that the state defined in (20) is normalized: \( \langle \alpha|\alpha \rangle = 1 \).

\(\star\) The coherent state \( |\alpha\rangle \) is an eigenket of the annihilation operator \( a \) with eigenvalue \( \alpha \); likewise \( \langle \alpha| \) is an eigenbra of the creation operator with eigenvalue \( \alpha^* \):

\[
 a|\alpha\rangle = \alpha|\alpha\rangle; \quad \langle \alpha|a^\dagger = \alpha^* \langle \alpha |
\]

(21)

\(\Box\) Exercise. Verify \( a|\alpha\rangle = \alpha|\alpha\rangle \) starting with the definition in (20).

- One can use (21) to find the matrix element of any product of powers of \( a \) and \( a^\dagger \) between two coherent states if they are in normal order: creation operators to the left of annihilation operators. Thus for \( p \) and \( q \) nonnegative integers,

\[
\langle \alpha| a^\dagger p a^q |\beta\rangle = (\alpha^*)^p \beta^q.
\]

(22)

\(\Box\) Exercise. Use (22) to show that

\[
\langle \alpha|\hat{X}|\alpha\rangle = \sqrt{2}\text{Re}(\alpha), \quad \langle \alpha|\hat{X}^2|\alpha\rangle = \frac{1}{\pi} + 2|\text{Re}(\alpha)|^2.
\]

(23)

\(\Box\) Exercise. Evaluate \( \langle \alpha|\hat{P}|\alpha\rangle \) and \( \langle \alpha|\hat{P}^2|\alpha\rangle \).

\(\star\) The coherent states \( |\alpha\rangle \) for different \( \alpha \) are not orthogonal to each other, unlike the number states \( |n\rangle \), or the states of the position representation, which are orthogonal in the formal sense of \( \langle x|x'\rangle = \delta(x-x') \).

\(\Box\) Exercise. Let \( \alpha \) and \( \beta \) be two complex numbers. Find an expression for \( \langle \alpha|\beta\rangle \). Show that its magnitude \( |\langle \alpha|\beta\rangle| \) depends only on the absolute value \( |\alpha - \beta| \) of the difference.

- Coherent states are **complete** in the sense that any \( |\psi\rangle \) in the Hilbert space can be written, at least formally, as a sum or integral over a collection of coherent states. However, this can be done in more than one way, so one says that the set of coherent states is **over complete**.

- One can express the identity operator \( I \) as an integral of coherent-state dyads in various different ways. Here is one that is sometimes useful:

\[
I = \frac{1}{\pi} \int_{-\infty}^{\infty} d\alpha' \int_{-\infty}^{\infty} d\alpha'' |\alpha\rangle\langle \alpha|,
\]

(24)
where \( \alpha' \) and \( \alpha'' \) are the real and imaginary parts of \( \alpha \).

\( \square \) Exercise. Justify (24) by showing that it defines an operator with the property that \( \langle n | I | n' \rangle = \delta_{nn'} \).

Hint: Evaluate the double integral \( \int d\alpha' \int d\alpha'' \) in polar coordinates.

\( \star \) The time dependence of a coherent state can be worked out starting with

\[
U(t)|n\rangle = e^{-i(n+1/2)\omega t} |n\rangle.
\] (25)

Thus

\[
U(t)|\alpha\rangle = e^{-i\omega t/2} \sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle = e^{-i\omega t/2} |\alpha(t)\rangle
\] (26)

if we define

\[
\alpha(t) := \alpha e^{-i\omega t}.
\] (27)

• That is to say, \( \alpha(t) \) traces out a circle in the complex plane, moving clockwise as \( t \) increases. Just like the elliptical orbit of the point representing a classical oscillator in the \( x, p \) phase space, which of course can be made into a circle by choosing appropriate units for \( x \) and \( p \).

\( \square \) Exercise. Let \( \alpha(t) = re^{-i\omega t} \), \( r > 0 \) a real number. Evaluate \( \langle \alpha(t)|X|\alpha(t)\rangle \) and \( \langle \alpha(t)|P|\alpha(t)\rangle \), and compare with what you would expect for a classical oscillator.

### 4.2 The displacement operator \( D(\alpha) \)

\( \star \) Define the displacement operator (the justification for this name will appear later)

\[
D(\alpha) = \exp[\alpha a^\dagger - \alpha^* a],
\] (28)

where \( \alpha \) is any complex number, and \( a \) and \( a^\dagger \) are the annihilation (lowering) and creation (raising) operators defined in Sec. 1.

• The operator \( D(\alpha) \) is unitary, as follows from the observation that the exponent in (28) is anti-Hermitian or skew Hermitian, which means it is equal to its adjoint times a minus sign:

\[
(aa^\dagger - \alpha^* a)^\dagger = \alpha^* a - \alpha a^\dagger = -(aa^\dagger - \alpha^* a).
\] (29)

Consequently,

\[
D^\dagger(\alpha) = \exp[-\alpha a^\dagger + \alpha^* a] = [D(\alpha)]^{-1} = D(-\alpha).
\] (30)

\( \star \) The coherent state \( |\alpha\rangle \) can be written as

\[
|\alpha\rangle = D(\alpha)|0\rangle.
\] (31)

• To prove this it is helpful to have handy a formula for manipulating exponentials of two noncommuting operators \( A \) and \( B \) which have the property that their commutator \([A, B]\) commutes with both \( A \) and \( B \):

\[
[A, [A, B]] = 0 = [B, [A, B]].
\] (32)

• Given this condition one can show (the argument is given in Townsend, p. 279, in the form of an exercise) that

\[
e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}.
\] (33)

• Let \( A = \alpha a^\dagger, B = -\alpha^* a \). Then \( [A, B] = |\alpha|^2 I \), which we will simply write as \( |\alpha|^2 \), and using (33) we see that \( D(\alpha) \) in (28) can be rewritten as

\[
D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a}.
\] (34)

We use this to evaluate

\[
D(\alpha)|0\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a} |0\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,
\] (35)
where the right side coincides with the earlier definition of $|\alpha|$ in (20).

- In writing (35) we have twice used the expansion $e^A = I + A + A^2/2! + \cdots$. When $e^{-\alpha^\ast a}$ is expanded in this way and applied to $|0\rangle$ we get $|0\rangle$, for any positive power of $a$ applied to $|0\rangle$ gives 0. The expansion of $e^{\alpha a^\dagger}$ applied to $|0\rangle$ gives the sum of the right side of (35) when one takes into account the $\sqrt{n!}$ that appears in (13).

□ Exercise. What is $D(\alpha)D(\beta)$ equal to? First make a guess (after all, if a displacement is followed by a second displacement, then . . . ). Next show that your guess is correct up to a phase factor, which you should evaluate.

4.3 Wave packet in position space

★ As the next step we note that the exponent in the definition of $D(\alpha)$ can be written, using (6), in the form

$$a\alpha^\dagger - \alpha^* a = i\sqrt{2}[\alpha'' \hat{X} - \alpha' \hat{P}],$$

where $\alpha = \alpha' + i\alpha'' = \text{Re}(\alpha) + i\text{Im}(\alpha)$.

□ Exercise. Check it.

- As a consequence, setting $A = i\sqrt{2}\alpha'' \hat{X}$ and $B = -i\sqrt{2}\alpha' \hat{P}$, and noting that $[\hat{X}, \hat{P}] = iI$, we can employ (33) to write

$$D(\alpha) = e^{-i\alpha'' \alpha' u} e^{i\sqrt{2}\alpha'' \hat{X}} e^{-i\sqrt{2}\alpha' \hat{P}}.$$  

(37)

★ We make use of (37) by noting that in the position representation, see (9), $\hat{P} = -i\partial/\partial u$, and hence

$$e^{-i\sqrt{2}\alpha' \hat{P}} = e^{-i\sqrt{2}\alpha' (\partial/\partial u)}$$

(38)

is the operator which applied to some (nice) function $f(u)$ yields the shifted function $f(u - \sqrt{2}\alpha')$. As a consequence (37) allows us to write down an explicit form for $|\alpha\rangle$ in the position representation:

$$\phi(\alpha; u) := \langle u|\alpha\rangle = e^{-i\alpha'' \alpha' u} e^{i\sqrt{2}\alpha'' u} \phi_0(u - \sqrt{2}\alpha').$$

(39)

- It is helpful to explore the significance of (39) starting with the case in which $\alpha' = 0$ is a positive real number. In that case, since $\alpha'' = 0$, (39) tells us that $\phi(\alpha; u)$ is simply the Gaussian function (14), the harmonic oscillator ground state, with its center shifted to the right by an amount $\sqrt{2}\alpha'$. Consequently, the mean of the corresponding probability distribution density for $u$, given by $|\phi(\alpha; u)|^2$ is $\sqrt{2}\alpha'$, and its standard deviation or “uncertainty” $\Delta u$ is identical to that of the ground state.

- This justifies the name “displacement operator” for $D(\alpha)$, at least in the case in which $\alpha$ is real: it simply displaces the position-space wave function.

- But is it not an embarrassment, or at least an annoyance, that the displacement is $\sqrt{2}\alpha'$ and not $\alpha'$? Indeed, but there is not much one can do about it. One could employ a complex number $\tilde{\alpha} = \sqrt{2}\alpha$ in place of $\alpha$, but then there are factors of $\sqrt{2}$ in the exponent defining the displacement operator in terms of $\tilde{\alpha}$. Or one could use a different definition for the characteristic length $\xi$ of a harmonic oscillator. One way or another, a $\sqrt{2}$ will creep into the discussion.

- Returning to (39). What happens if $\alpha''$, the imaginary part of $\alpha$ is nonzero? First there is the phase factor $e^{-i\alpha'' \alpha' u}$, which has no effect on the physics represented by this state, so we will ignore it.

- If one is considering (which at the moment we are not) linear combinations of coherent states having different values of $\alpha$, then one must pay some attention to these initial phases.

- The factor $e^{i\sqrt{2}\alpha'' u}$, on the other hand, is a phase which depends upon the position $u$, so it has physical significance: it indicates that in addition to having a displaced position (assuming $\alpha' \neq 0$) the wave packet also has some momentum associated with it; that is, it represents a particle which is moving.

□ Exercise. Work out the corresponding description of a wave packet in momentum space using the analog of (37), but with the $\hat{P}$ and $\hat{X}$ exponentials in the opposite order.

★ As a function of time $\alpha$ moves clockwise on a circle in the complex plane. Let us suppose that

$$\sqrt{2}\alpha = \sqrt{2}\alpha' + i\sqrt{2}\alpha'' = b(\cos \tau - i \sin \tau),$$

where $\tau := \omega t$.

(40)
is the dimensionless time.

• Then aside from an overall phase, the position-space wave packet is
  \[ \phi(u, \tau) = e^{i(-b \sin \tau)u} \phi_0(u - b \cos \tau), \]  
where remember that \( \phi_0(u), (14) \), is a Gaussian centered at \( u = 0 \).

• The explicit function given in (41) provides a nice way of visualizing what happens as (dimensionless) time \( \tau \) increases. The probability distribution density for position,
  \[ \rho(u, \tau) = |\phi(u, \tau)|^2 = | \phi_0(u - b \cos \tau)|^2, \]  
is that of the ground state shifted by a distance \( b \cos \tau \). So one can think of the wave packet as oscillating back and forth while maintaining its Gaussian shape.

• In addition there is a position-dependent phase \( e^{i(-b \sin \tau)u} \) which tells us that the (dimensionless) momentum \( v \) has an average value of \( -b \sin \tau \) corresponding to a nonzero average velocity. This is zero when \( \tau \) is a multiple of \( \pi \), i.e., when the (average) position \( u \) reaches its maximum or minimum value. Similarly, when \( \tau \) is \( \pi/2 \) plus an integer times \( \pi \), the (average) position is zero, and the momentum or velocity reaches its maximum or minimum value. This corresponds quite well with the motion of a classical oscillator.

• Of course, neither position nor momentum is precisely defined in a coherent state, and in dimensionless units the associated uncertainties are of the order of 1. Taking this into account one can visualize the wave packet, roughly speaking, as a circular orbit in the classical phase space, but “smudged out” a bit in both position and momentum.

• In the case in which \( b \gg 1 \) this “smudging out” is a relatively minor effect in comparison with the overall motion, and the classical picture becomes a quite good approximation to the behavior of the quantum oscillator in a coherent state.

• The nice correspondence between quantum unitary time development and a classical orbit is special to the harmonic oscillator. For other potentials, for example a square well or a Coulomb potential, unitary time development is not well approximated by a classical orbit, and the Ehrenfest relations, while they remain formally correct, can be quite misleading.

★ The probability current in dimensionless form corresponding to (41) is given by
  \[ j = \frac{\partial}{\partial x} \right\| \phi \right\|^2 = - (b \sin \tau) | \phi_0(u - b \cos \tau)|^2. \]  

• Thus for a coherent state (but not for a general harmonic oscillator state) the probability current at a particular time is proportional to the probability distribution density (42). This reflects the fact that the latter moves “rigidly” as time progresses, i.e., without changing its shape.

□ Exercise. Check that the (dimensionless) probability conservation equation
  \[ \frac{\partial \rho}{\partial t} = - \frac{\partial j}{\partial x} \]  
is satisfied when \( \rho \) is given by (42) and \( j \) by (43).