1 States

1.1 Introduction

In quantum theory, identical particles come in two varieties: bosons and fermions. The wave function for a collection of identical bosons must be symmetrical under the interchange of any two particles. Such wave functions lie in a Hilbert space $H^S$ of symmetrical functions, which is a subspace of the tensor product of single-particle states (orbitals). The wave function for a collection of identical fermions must be antisymmetrical, that is, it must change sign under the interchange of any two particles. These wave functions lie in a Hilbert space $H^A$ of antisymmetrical functions, which is another subspace of the tensor product of single-particle states.

1.2 Orbitals

Let $\mathcal{F}$ be the Hilbert space of a single particle. Its elements are single-particle states or orbitals; these must be carefully distinguished from the many-particle states discussed below.

- An orbital can be represented by a ket $|\alpha\rangle$ or by a wavefunction
  \[ \alpha(q) = \langle q | \alpha \rangle, \]
  where $q$ represents the set of coordinates corresponding to a single particle. For example, for a spin-less particle in one dimension, $q$ is just a single real number, the position. For a spin half particle in three dimensions,
  \[ q = (r, \sigma) = (x, y, z, \sigma), \]
  where $x, y, \text{ and } z$ are the Cartesian coordinates and $\sigma = \pm 1$ is the spin coordinate.

- The inner product of two orbitals is
  \[ \langle \alpha | \beta \rangle = \int \alpha^*(q) \beta(q) \, dq, \]
  where $dq$ denotes an integral over the continuous coordinates and a sum over the spin coordinate.
1.3 Tensor product space

- To discuss a system of \( N \) identical particles, we first construct the Hilbert space
  \[ \mathcal{H} = \mathcal{F} \otimes \mathcal{F} \otimes \cdots \mathcal{F} \]
  of states without particle exchange symmetry, using \( N \) copies of \( \mathcal{F} \).

  - An element of \( \mathcal{H} \) is a complex valued function of \( N \) arguments, each a coordinate set for a single particle:
    \[ \psi(q_1, q_2, \ldots, q_N). \]

  - The order of the arguments is important. For example, if we are considering 3 spinless particles in 1 dimension, we might have:
    \[ \psi(u, v, w) = (u + 2v^2 + 3w^3) \exp[-u^2 - v^2 - w^2]. \]
    Then
    \[ \psi(q_1, q_2, q_3) = (q_1 + 2q_2^2 + 3q_3^3) \exp[-q_1^2 - q_2^2 - q_3^2], \]
    is not the same function as
    \[ \psi(q_2, q_1, q_3) = (q_2 + 2q_1^2 + 3q_3^3) \exp[-q_1^2 - q_2^2 - q_3^2]. \]

- The inner product of two elements of \( \mathcal{H} \) is defined by
  \[ \langle \phi | \psi \rangle = \int \phi^*(q_1, q_2, \ldots, q_N) \psi(q_1, q_2, \ldots, q_N) \, dq_1 \, dq_2 \ldots \, dq_N. \]

1.4 Permutations

- Permutations. A permutation \( P \) of the integers \( \{1, 2, \ldots, N\} \) is a function which maps each of these integers to another integer in the same collection in a one-to-one manner.

  - For example, with \( N = 3 \),
    \[ P(1) = 3, \quad P(2) = 1, \quad P(3) = 2 \]
    is a permutation.

  - There are \( N! \) permutations, including the identity \( I \), which maps every integer to itself. These permutations form a group under composition in the sense that if \( P \) and \( Q \) are permutations, so is \( QP = Q \circ P \),
    \[ (QP)(j) = Q(P(j)). \]
    The inverse permutation of \( P \) is \( P^{-1} \);
    \[ PP^{-1} = P^{-1}P = I. \]

  - An interchange is a permutation which exchanges two integers and leaves the others fixed. For example,
    \[ P(1) = 3, \quad P(2) = 2, \quad P(3) = 1 \]
    is an interchange, as it exchanges 1 and 3.

  - Any permutation can be written as a product of interchanges. If the number of interchanges in the product is zero or an even number, the permutation is even, if the number of interchanges is odd, the permutation is odd.

- With every permutation \( P \) of \( \{1, 2, \ldots, N\} \) we associate a permutation operator \( \hat{P} \) on \( \mathcal{H} \) that maps a function \( \psi \) to a function \( \hat{P}\psi \) defined by
  \[ (\hat{P}\psi)(q_1, q_2, \ldots, q_N) = \psi(q_{P(1)}, q_{P(2)}, \ldots, q_{P(N)}). \]
For example, if $P$ is the permutation in (10),
\[
\hat{P}\psi(q_1, q_2, q_3) = \psi(q_3, q_1, q_2).
\] (15)

- If $Q$ and $P$ are two permutations, the product of the corresponding operators corresponds to the composition of the permutations:
\[
(\hat{Q}\hat{P}\psi)(q_1, q_2, \ldots, q_N) = \psi(q_{Q(P(1))}, q_{Q(P(2))}, \ldots, q_{Q(P(N))}).
\] (16)

- The adjoint of $\hat{P}$ is the permutation operator defined by the inverse permutation $P^{-1}$, and thus $\hat{P}^\dagger = \hat{P}^{-1}$. Consequently, each permutation operator is a unitary operator.
- The permutation operators form a group under multiplication which is isomorphic to the group of permutations under composition.

### 1.5 Symmetric and antisymmetric spaces

- We define the symmetric $S$ and antisymmetric projectors on $\mathcal{H}$ by means of the formulas:
\[
S = \frac{1}{N!} \sum_P \hat{P},
\] (17)
\[
A = \frac{1}{N!} \sum_P (-1)^P \hat{P},
\] (18)
where the sums are over all $N!$ permutations, and $(-1)^P$ is +1 if $P$ is an even permutation, and −1 if $P$ is odd.

- If $Q$ is any permutation, one can show that
\[
\hat{Q}S = S\hat{Q} = S, \quad \hat{Q}A = A\hat{Q} = (-1)^Q A.
\] (19)

Using this, one can show that
\[
S = S^\dagger = S^2, \quad A = A^\dagger = A^2, \quad SA = 0 = AS.
\] (20)

Thus $S$ and $A$ are projectors onto mutually orthogonal subspaces $\mathcal{H}^S$ and $\mathcal{H}^A$ of $\mathcal{H}$. Each subspace is itself a Hilbert space; $\mathcal{H}^S$ is the space of symmetrical functions appropriate for describing identical bosons, and $\mathcal{H}^A$ is the space of antisymmetrical functions appropriate for describing identical fermions.

- Given any permutation operator $\hat{Q}$, and any element $\psi^S$ of $\mathcal{H}^S$, (19) implies that
\[
\hat{Q}\psi^S = \psi^S,
\] (21)
and likewise, if $\phi^A$ is any element of $\mathcal{H}^A$,
\[
\hat{Q}\phi^A = (-1)^Q \phi^A.
\] (22)

- For example, since an interchange is an odd permutation, for $N = 3$ we have
\[
\psi^A(q_2, q_1, q_3) = -\psi^A(q_1, q_2, q_3).
\] (23)

### 1.6 Configurations

- Let $\{ |\nu\rangle \}$ be a collection of orbitals which form an orthonormal basis of $\mathcal{F}$. (It is not essential that they be orthonormal, but this simplifies the treatment which follows.) Particular orbitals will be denoted by $|\alpha\rangle$, $|\beta\rangle$, etc.

- A rather natural way to construct symmetrical or antisymmetrical states is to start with a state in $\mathcal{H} = \mathcal{F}^N$ which is a product of orbitals, and apply to it $S$ or $A$. 

3
• For example, with \( N = 3 \), let

\[
\psi(q_1, q_2, q_3) = \alpha(q_1)\alpha(q_2)\beta(q_3) = \langle q_1, q_2, q_3|\alpha\beta\rangle.
\]  

(24)

That is, particles 1 and 2 are in the same orbital \( \alpha \), but particle 3 is in a different orbital \( \beta \) which is orthogonal to \( \alpha \). Then write

\[
\psi^S(q_1, q_2, q_3) = S\psi(q_1, q_2, q_3) = \frac{1}{3} \left[ \alpha(q_1)\alpha(q_2)\beta(q_3) + \alpha(q_1)\alpha(q_3)\beta(q_2) + \alpha(q_2)\alpha(q_3)\beta(q_1) \right].
\]

(25)

This function, which is invariant under any permutation operator, is a suitable state to describe bosons.

• We get precisely the same result for \( \psi^S \) if we apply \( S \) to an initial product state which involves the same collection of orbitals but assigns particles to them in a different way. Thus if

\[
\psi(q_1, q_2, q_3) = \alpha(q_1)\beta(q_2)\alpha(q_3),
\]

(26)

it is easy to check that \( S\bar{\psi} = S\psi \).

○ If we apply \( A \) to \( \psi \) as defined in (24), the result is:

\[
A\psi(q_1, q_2, q_3) = 0.
\]

(27)

This cannot describe a state of identical fermions.

• In order that \( \psi^A = A\psi \) not vanish, it is necessary that \( \psi \) be a product of \( N \) distinct orbitals. This is the Pauli exclusion principle: no more than one fermion in any orbital.

• If both \( \psi \) and \( \bar{\psi} \) are product states based upon the same collection of distinct orbitals, but assigning different particles to the different orbitals, then \( A\psi = \pm A\bar{\psi} \). Since a phase factor makes no difference to the physical interpretation, both of these stand for the same physical state.

○ In order to specify a state in \( \mathcal{H}^S \) corresponding to a certain collection of orbitals, we only need to know which orbitals fall in this collection, and how many particles are in each orbital. We do not have to know which particle is in which orbital, since the projector \( S \) will move them around anyway. This makes physical sense: if the particles are identical, surely it makes no difference which particle we call 1 and which we call 2, etc.

• Hence, instead of dealing with a big, messy wave function containing \( N! \) terms, it suffices to simply provide a list of orbitals, and an indication of how many particles are in each one. Thus, rather than writing out \( \psi^S \) explicitly, as in (25), we can use the compact form

\[
|\alpha^2\beta, S\rangle, \text{ or } \langle q_1, q_2, q_3|\alpha^2\beta, S\rangle.
\]

(28)

Here \( \alpha^2 \) indicates that there are two particles in orbital \( \alpha \), \( \beta \) that there is one particle in orbital \( \beta \). Empty orbitals are omitted from the notation. The label \( S \) indicates that this is a symmetrical state belonging to \( \mathcal{H}^S \); if it is clear from the context that we are dealing with a symmetrical state, the \( S \) can be omitted.

• It will be convenient to assume that the ket \( |\alpha^2\beta, S\rangle \) is normalized,

\[
\langle \alpha^2\beta, S|\alpha^2\beta, S\rangle = 1,
\]

(29)

where note that the order of the symbols in the bra \( (\langle \) is the same as in the ket \( |) \). We shall also assume a particular phase for \( |\alpha^2\beta, S\rangle \). It is determined by requiring that

\[
\langle q_1, q_2, q_3|\alpha^2\beta, S\rangle = c\alpha(q_1)\alpha(q_2)\beta(q_3) + \ldots,
\]

(30)

with \( c \) a positive constant.

○ Note, by the way, that \( \langle q_1, q_2, q_3|\alpha^2\beta, S\rangle \) differs from \( \psi^S(q_1, q_2, q_3) \) in (25) in that the latter is not normalized.

• Two states based on different occupation numbers for different orbitals are orthogonal to each other. Thus \( |\alpha^2\beta, S\rangle \) is orthogonal to \( |\alpha\beta^2, S\rangle \) and to \( |\alpha\beta\gamma, S\rangle \), etc.
The state $|\alpha^2 \beta, S\rangle$, and others like it, will be referred to as a (symmetrical) configuration of the system of $N$ identical particles. Since product states form a basis of $\mathcal{H} = \mathcal{F}^N$, it follows that the configurations based upon a particular set (basis) of orbitals in $\mathcal{F}$ form a basis of $\mathcal{H}^S$.

Note that this does not mean that every state in $\mathcal{H}^S$ is a configuration. It does mean that any state in $\mathcal{H}^S$ can be written as a linear combination of configurations.

What constitutes a configuration depends, of course, upon the choice of orbitals which provide the basis of $\mathcal{F}$. Different choices lead to different possibilities.

Two configurations are orthogonal if they involve different sets of orbitals. Thus $|\alpha^2 \beta, S\rangle$ is orthogonal to $|\alpha \beta^2, S\rangle$. But note that $|\alpha^2 \beta, S\rangle$ and $|\beta \alpha^2, S\rangle$ are simply two different ways of writing the same thing: the configuration in which there are two particles in orbital $\alpha$ and one in orbital $\beta$.

Antisymmetric configurations are defined in the same way, except that, of course, one has to start with a product of $N$ different orbitals in $\mathcal{H}$ to produce a corresponding non-zero state in $\mathcal{H}^A$. For example, with $N = 3$, $|\alpha \beta \gamma, A\rangle$ means a configuration in which there is one particle in orbital $\alpha$, one in $\beta$, and one in $\gamma$.

We assume the configuration is normalized,

$$\langle \alpha \beta \gamma, A | \alpha \beta \gamma, A \rangle = 1,$$

and the phase is chosen such that

$$(q_1, q_2, q_3 | \alpha \beta \gamma, A) = c \alpha(q_1) \beta(q_2) \gamma(q_3) \pm \ldots,$$

with $c$ a positive constant. As a consequence of this sign convention, the order of symbols inside a ket does influence the phase:

$$|\alpha \beta \gamma, A\rangle = -|\beta \alpha \gamma, A\rangle = +|\gamma \alpha \beta, A\rangle,$$

and so forth. Changing the sign does not influence the physical significance of the configuration, of course. On the other hand, it does make a difference when constructing states that are linear combinations of different configurations. For example,

$$|\psi_a\rangle = |\alpha \beta \gamma, A\rangle + |\alpha \beta \delta, A\rangle, \quad |\psi_b\rangle = |\beta \alpha \gamma, A\rangle + |\alpha \beta \delta, A\rangle,$$

are two different states in $\mathcal{H}^A$!

Note on terminology. In atomic physics the terms “configuration” and “orbital” have a technical definition which differs somewhat from the way they are employed here, although the basic idea is similar.

## 2 Operators

### 2.1 Introduction

Physical properties for systems of identical fermions correspond to subspaces of $\mathcal{H}^A$, or the corresponding projectors. Physical quantities are represented by Hermitian operators which map $\mathcal{H}^A$ to itself. Unitary time development is given, as usual, by a family of unitary operators $T(t', t)$ which map $\mathcal{H}^A$ to itself. The same comments apply to a collection of identical bosons, with $\mathcal{H}^S$ replacing $\mathcal{H}^A$. Thus there is nothing distinctive about the quantum formalism of identical particles, once the correct Hilbert space has been identified.

However, a common procedure is to start with operators defined on the (unsymmetrized) tensor product space $\mathcal{H}$, (4), and then construct their counterparts on $\mathcal{H}^A$ or $\mathcal{H}^S$, as the case may be. For example, the Hamiltonian operator for a (non-relativistic) many-electron atom is of the form

$$H = B + C,$$

where

$$B = \sum_j b_j$$
is a one-body or single-particle operator because it is a sum of terms

$$b_j = -\frac{\hbar^2}{2m} \nabla_j^2 - \frac{Ze^2}{4\pi\epsilon_0|r_j|},$$

(37)

each of which refers to a single electron, while

$$C = \sum_{j \neq k} c_{jk}$$

(38)

is a two-body or two-particle operator because each term

$$c_{jk} = \frac{e^2/2}{4\pi\epsilon_0|r_j - r_k|}$$

(39)

in the sum refers to two electrons.

- Both $B$ and $C$ as defined here are operators on $\mathcal{H}$. However, they have the property that they are symmetrical, they do not “distinguish” the different particles: the mass of every electron is the same as that of very other electron, the Coulomb repulsion between electrons 2 and 3 is of precisely the same functional form as that between electrons 1 and 4, etc. This symmetry is expressed formally through the fact that $B$ and $C$ commute with all the permutation operators defined in (14). Consequently, they also commute with the projectors $S$ and $A$ used in defining $\mathcal{H}^S$ and $\mathcal{H}^A$. This means that $B$ maps $\mathcal{H}^A$ into itself and $\mathcal{H}^S$ into itself, and the same is true of $C$.

- Even when one knows the definition of such symmetrical operators on $\mathcal{H}$, it is not immediately obvious what effect they will have on some configuration in $\mathcal{H}^A$ or $\mathcal{H}^S$. The purpose of the following two sections of these notes is to provide some insight into this matter. The goal is not to find efficient rules of computation. The latter are best obtained using the formalism of “second quantization.”

### 2.2 One-body operators

- A (symmetrical) one-body operator on $\mathcal{H} = \mathcal{F}^N$ takes the form

$$B = b \otimes I \cdots \otimes I + I \otimes b \otimes \cdots \otimes I + \cdots I \otimes I \cdots \otimes b,$$

(40)

where the prototype operator $b$ can be any operator on the Hilbert space $\mathcal{F}$ of a single particle. If, in particular, the orbitals $\{|\nu\rangle\}$ form an orthonormal basis of $\mathcal{F}$, one can write $b$ as a sum of dyads,

$$b = \sum_{\mu \nu} \langle \mu | b | \nu \rangle \cdot |\mu\rangle\langle \nu|,$$

(41)

with complex coefficients.

- For purposes of illustration, suppose that $N = 3$, and that $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$, and $|\delta\rangle$, are distinct orbitals, and use a notation in which

$$|\alpha\beta\gamma\rangle = |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle$$

(42)

is the (unsymmetrized) state in $\mathcal{H} = \mathcal{F}^3$ in which particle 1 is in $\alpha$, particle 2 is in $\beta$, and particle 3 is in $\gamma$. In this notation, the corresponding antisymmetrical state is

$$|\alpha\beta\gamma, A\rangle = \left\{ \frac{|\alpha\beta\gamma\rangle - |\alpha\gamma\beta\rangle + |\beta\gamma\alpha\rangle - |\beta\alpha\gamma\rangle + |\gamma\alpha\beta\rangle - |\gamma\beta\alpha\rangle}{\sqrt{6}} \right\}.$$

(43)

- To understand the action of $B$ on $|\alpha\beta\gamma, A\rangle$, it helps to consider various special cases. One can then work out the result for a more general $b$ by using linearity and adding up the terms in (41). Suppose, for example, that

$$b = |\beta\rangle\langle \beta|,$$

(44)

so that (40) takes the form

$$B = |\beta\rangle\langle \beta| \otimes I \otimes I + I \otimes |\beta\rangle\langle \beta| \otimes I + I \otimes I \otimes |\beta\rangle\langle \beta|.$$

(45)
It is then obvious that
\[ B|\alpha\beta\gamma, A\rangle = |\alpha\beta\gamma, A\rangle, \] (46)
because when (45) is applied to any of the terms on the right side of (43), say to \(|\gamma\alpha\beta\rangle\), two of the three operators in (45) give zero, but the remaining one gives \(|\gamma\alpha\beta\rangle\).

- Suppose that in place of (44), we use
\[ b = |\delta\rangle\langle\beta|. \] (47)
Then the same sort of analysis—replace \(|\beta\rangle\langle\beta|\) in (45) with \(|\delta\rangle\langle\beta|\)—shows that
\[ B|\alpha\beta\gamma, A\rangle = |\alpha\delta\gamma, A\rangle; \] (48)
that is, \(\beta\) is replaced by \(\delta\) inside the ket. It is important to do the replacement without changing the positions of any of the other symbols, because the sign of an antisymmetric configuration depends upon the order of the symbols.

- Of course, if we let
\[ b = |\beta\rangle\langle\beta|, \] (49)
then \(B\) applied to \(|\alpha\beta\gamma, A\rangle\) gives zero, as the orbital \(\delta\) is not occupied in this configuration. However, note that one also gets zero if
\[ b = |\gamma\rangle\langle\beta|, \] (50)
because the effect upon \(|\alpha\beta\gamma, A\rangle\) would be to put two particles in orbital \(\gamma\), and Pauli would object. One can see that if the \(B\) corresponding to the \(b\) in (50) is applied to the right side of (43), the individual terms do not vanish, but they cancel one another when one adds them up.

Similar results are obtained for \(N = 3\) identical bosons, but the details are somewhat different. Thus if \(b = |\beta\rangle\langle\beta|\), (44), we again get
\[ B|\alpha\beta\gamma, S\rangle = |\alpha\beta\gamma, S\rangle, \] (51)
as in (46). However, if there is more than one particle in the \(\beta\) orbital, this shows up as a numerical coefficient which is no longer 1:
\[ B|\alpha\beta^2, S\rangle = 2|\alpha\beta^2, S\rangle. \] (52)

- If \(b = |\delta\rangle\langle\beta|\), (47), then (48) is correct with \(A\) replaced by \(S\). On the other hand,
\[ B|\alpha\beta\delta, S\rangle = \sqrt{2}|\alpha\beta^2, S\rangle, \] (53)
in contrast to the case of fermions, where the right side would be 0.

- Working out the rules for the numerical coefficients of the sort which appear on the right sides of (52) and (53) is beyond the scope of these notes. It is done in detail in books on many-particle quantum theory, such as Fetter and Walecka\(^1\).

### 2.3 Two-body operators

We assume that a prototype operator \(c\) is defined on the two-particle space \(\mathcal{F} \otimes \mathcal{F}\), and use this to define the symmetrical operator
\[ C = \sum_{j \neq k} c_{jk}, \] (54)
where \(c_{jk}\) is \(c\) acting on vectors in the space \(\mathcal{F}_j \otimes \mathcal{F}_k\) of particles \(j\) and \(k\). We do not assume that \(c_{jk} = c_{kj}\), but it is possible to rewrite (54) as a sum
\[ C = \sum_{j < k} (c_{jk} + c_{kj}) \] (55)
of symmetrical combinations. Consequently, \(C\) commutes with all the permutation operators on \(\mathcal{H}\).

\(^1\)Quantum Theory of Many-Particle Systems (1971).
As before, we assume an orthonormal basis \{|\nu\rangle\} of \mathcal{F}. Using this, the prototype \(c\) can be written as a sum
\[
c = \sum_{\bar{\mu},\bar{\nu}} \sum_{\mu,\nu} \langle \bar{\mu} \bar{\nu} | c | \mu \nu \rangle \cdot | \bar{\mu} \bar{\nu} \rangle \langle \mu \nu |
\]
where note that one can also write
\[
| \bar{\mu} \bar{\nu} \rangle \langle \mu \nu | = | \bar{\mu} \rangle \langle \mu | \otimes | \bar{\nu} \rangle \langle \nu |.
\]

Once again, it helps to look at the case where the prototype is a single dyad, say
\[
c = | \bar{\alpha} \bar{\beta} \rangle \langle \alpha \beta |
\]
It is easy to convince oneself that the general effect of the corresponding \(C\) acting on an orbital will be to remove one particle from orbital \(\alpha\) and one particle from orbital \(\beta\), and to add a particle to orbital \(\bar{\alpha}\) and a particle to \(\bar{\beta}\). However, there may also be a numerical factor (possibly zero) associated with the procedure.

To work out the action of \(C\) corresponding to the prototype \(c\) in (58) is relatively straightforward in the case of fermions, \(H^A\). If one assumes that \(\alpha, \beta, \bar{\alpha}, \text{and} \bar{\beta}\) are all distinct, then \(C\) gives zero when applied to a configuration unless \(\alpha\) and \(\beta\) are both occupied, and \(\bar{\alpha}\) and \(\bar{\beta}\) are both empty. When both of these conditions are fulfilled, the action of \(C\) is given by:
\[
C | \cdots \alpha \cdots \beta \cdots , A \rangle = | \cdots \bar{\alpha} \cdots \bar{\beta} \cdots , A \rangle.
\]
That is, \(\alpha\) is replaced by \(\bar{\alpha}\), and \(\beta\) by \(\bar{\beta}\) in the list of occupied orbitals, with all the other symbols remaining the same.

The case where the four orbitals are not all distinct, say \(\bar{\alpha} = \alpha, \beta, \bar{\beta}\) three different orbitals, can be worked out by the same methods. While it is relatively straightforward, the task of working out this and the other remaining special cases is a trifle tedious, and it is for this reason that the second quantization formalism is preferred if one wants to carry out a serious calculation.

In the case of bosons, \(H^S\), the effect of \(C\) corresponding to the prototype \(c\) in (58) is the same as for fermions, (59) with \(A\) replaced by \(S\), in the case in which \(\alpha, \beta, \bar{\alpha}, \text{and} \bar{\beta}\) are all distinct, and in which there is, initially, one particle in \(\alpha\) and \(\beta\), and there are no particles in \(\bar{\alpha}\) and \(\bar{\beta}\). Also, unless there is at least one particle in \(\alpha\) and at least one particle in \(\beta\), \(C\) gives zero. However, there is now no restriction on the number of particles initially in \(\bar{\alpha}\) and \(\bar{\beta}\), and the numerical coefficient will depend upon these numbers, as well as the numbers of particles (if more than one) initial present in \(\alpha\) and \(\beta\).

In addition, prototype operators of the form
\[
| \bar{\alpha} \bar{\beta} \rangle \langle \alpha \alpha |, \quad | \bar{\alpha} \bar{\alpha} \rangle \langle \alpha \beta |, \quad | \bar{\alpha} \bar{\alpha} \rangle \langle \alpha \alpha |
\]
need to be considered in the case of bosons. (For fermions, the operator \(C\) corresponding to any of these gives zero for all vectors in \(H^A\).) The effect of \(C\) for the first of these possibilities is to reduce by two the number of particles in orbital \(\alpha\), and add one particle to \(\bar{\alpha}\), and one to \(\bar{\beta}\). If the occupancy of \(\alpha\) is initially 0 or 1, \(C\) yields zero when applied to this configuration. Similar comments apply to the other possibilities in (60).

The numerical factors in cases in which \(C\) yields a non-zero result are outside the scope of these notes.