Entanglement vs Discord: Who Wins?

Vlad Gheorghiu

Department of Physics
Carnegie Mellon University
Pittsburgh, PA 15213, U.S.A.

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Reference: A. Brodutch and D. Terno, PRA **83**, 010301 (Rapid) (2011)
A summary of this talk is available online at
http://quantum.phys.cmu.edu/QIP

Quantifying (quantum) correlations

- Mutual Information (Classical):

$$I(A:B) = H(A) + H(B) - H(A,B)$$

- Yet another equivalent definition (Classical)

$$I(A:B) = H(B) - H(B|A)$$

- How does one think operationally about a conditional entropy?

$$\begin{aligned} H(B|A) &= \sum_a Pr(a) H(B|A=a) \\ &= -\sum_a Pr(a) \sum_b Pr(b|a) \log(Pr(b|a)) \\ &= -\sum_{a,b} Pr(a,b) \left[\log(Pr(a,b)) - \log(Pr(a))\right] \\ &= H(A,B) - H(A) \end{aligned}$$

- Quantum domain: replace H by S and classical random variables by quantum states!
- How to define the quantum conditional entropy?
- There are 2 ways:
  1. A simple way (but which lacks a nice operational interpretation, and can also be negative!!!)

$$S(B|A) = S(A,B) - S(A)$$

  2. An operational way, which involves measurements $\{\Pi_A^a\}$ on Alice's side:

$$S(B|\Pi_A) = \sum_a Pr(a) S(\rho_B^a), \quad Pr(a)\rho_B^a = \mathrm{Tr}_A\left(\Pi_A^a \otimes I_B \rho_{AB} \Pi_A^a \otimes I_B\right)$$

  and is always positive.

- They give rise to 2 (inequivalent) ways of writing the quantum mutual information:
  1. $S(A:B) = S(B) - S(B|A) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$, and
  2. $\chi(A:B,\Pi_A) = S(B) - S(B|\Pi_A) = S(\rho_B) - \sum_a Pr(a) S(\rho_B^a)$, with a choice of $\{\Pi_A^a\}$ in mind.
- The second definition is not symmetric and depends on who's doing the measurement and also on the measurement operators!
- Quantum discord (introduced by H. Ollivier and W. H. Zurek in PRL **88**, 017901 (2001)):

$$D^A(\rho_{AB},\Pi_A) = S(A:B) - \chi(A:B,\Pi_A).$$

- Sometimes an optimization is made over all possible measurement strategies

$$D^A(\rho_{AB}) = S(A:B) - \max_{\{\Pi_A\}} \chi(A:B,\Pi_A).$$

Properties of Quantum Discord

- In a sense, discord "removes" the classical correlations from a bipartite quantum state.
- $D^A(\rho_{AB}) \geqslant 0$
- $D^A(\rho_{AB}) = 0$ if and only if there exists an orthonormal basis $\{|\psi_k\rangle\}$ on Alice's side such that

$$\rho_{AB} = \sum_k p_k |\psi_k\rangle \langle\psi_k|_A \otimes \rho_{k\,B},$$

  see A. Datta, arXiv:0807.4490 [quant-ph].
- Recently Vedral et al proved in PRL **105**, 190502 (2010) that $D^A(\rho_{AB}) = 0$ if and only if

$$[L_n, L_m] = 0, \forall n, m,$$

  where $\rho_{AB} = \sum_n c_n L_n \otimes R_n$ is a Schmidt operator form.

- There are separable states that have non-zero quantum discord!
- Example:

$$\rho_{AB} = \frac{1}{4}[|0\rangle\langle 0| \otimes |+\rangle\langle +| + |1\rangle\langle 1| \otimes |-\rangle\langle -| + \\ + |+\rangle\langle +| \otimes |1\rangle\langle 1| + |-\rangle\langle -| \otimes |0\rangle\langle 0|].$$

- Hence discord is not an LOCC monotone!
- Zero discord environment states is a necessary and sufficient condition for CPTP evolution!

Entanglement vs Discord

Definition

A bilocal implementation G of a gate U on some finite set of separable states $\mathcal{L} = \{\rho_i^{in}\}_{i=1}^N$ (and their convex combinations) is a CPTP map that is implemented by local operations on the subsystems A and B, assisted by unlimited classical communication such that for any state $\rho_i^{in} \in \mathcal{L}$

$$G(\rho_i^{in}) = \sum_k K_k \rho_i^{in} K_k^\dagger \equiv U \rho_i^{in} U^\dagger = \rho_i^{out}$$

- The dual map $G^\dagger(\rho) := \sum_k K_k^\dagger \rho K_k$ satisfies

$$G^\dagger(\rho_i^{out}) = \rho_i^{in}$$

  for all *pure input states* $\rho_i^{in} \in \mathcal{L}$. Why?
- $\rho_i^{out} = G(\rho_i^{in}) = U \rho_i^{in} U^\dagger$ is pure, hence

$$1 = \langle \rho_i^{out} | \rho_i^{out} \rangle = \langle G(\rho_i^{in}) | \rho_i^{out} \rangle = \langle \rho_i^{in} | G^\dagger(\rho_i^{out}) \rangle$$

- If the set $\mathcal{L}$ is locally distinguishable, then the gate can be implemented by LOCC (obvious).
- If the action creates entanglement, then again the implementation must fail!
- However, the absence of entanglement is not sufficient!
- If one restricts the local operations to projective measurements and unitaries, then zero discord becomes a necessary criterion for such implementation success.
- That's because local measurements on a state of non-zero discord increases it's entropy, see PRA **81**, 062103 (2010), whereas the gate U does not!

- Result 1: If a set $\mathcal{L}$ contains one product state ($|00\rangle$) and the maximally mixed state $(I \otimes I)/4$, and the action of U is realized by LOCC, then all other allowed inputs (and their arbitrary convex combinations) must have zero discord!
- Example: the tiles

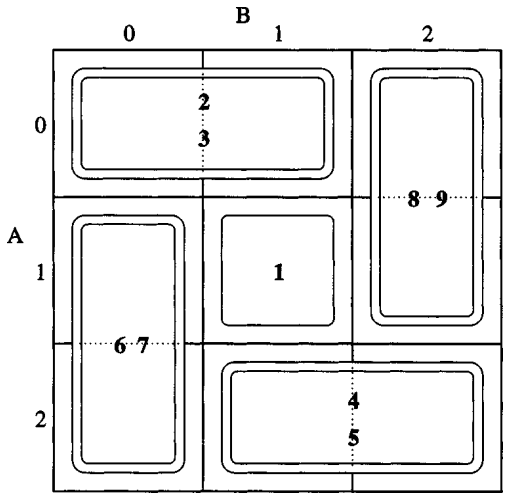

Figure: Bennett et al states

- Result 2: If the set $\mathcal{L}$ contains two pure non-orthogonal states, and the unitary operation is such that $D(\rho) \neq D(U\rho U^\dagger)$, where $\rho = p|\psi_1\rangle\langle\psi_1| + (1-p)|\psi_2\rangle\langle\psi_2|$, for some $0 < p < 1$, then it cannot be implemented on $\mathcal{L}$ by LOCC alone.

Conclusions:

- The absence of entanglement in both input and output does not automatically enable a remote implementation by LOCC.
- A discrepancy between local and global information content of separable states (which is captured by the discord) requires entanglement for their processing.
- Entanglement is required for any gate which changes the discord of the states!
- Recent results (arXiv:1006.4402, arXiv:1006.2460) suggest that a change in discord rather than entanglement is the required resource in computational speedup.