Entanglement vs Discord: Who Wins?

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1 Introduction to Quantum Discord

2 Entanglement vs Discord

Reference: A. Brodutch and D. Terno, PRA 83, 010301 (Rapid) (2011)
A summary of this talk is available online at http://quantum.phys.cmu.edu/QIP
Quantifying (quantum) correlations

- Mutual Information (Classical):
  \[ I(A : B) = H(A) + H(B) - H(A, B) \]

- Yet another equivalent definition (Classical):
  \[ I(A : B) = H(B) - H(B|A) \]

- How does one think operationally about a conditional entropy?
  \[
  H(B|A) = \sum_a Pr(a)H(B|A = a) \\
  = -\sum_a Pr(a)\sum_b Pr(b|a)\log(Pr(b|a)) \\
  = -\sum_{a,b} Pr(a, b)[\log(Pr(a, b)) - \log(Pr(a))] \\
  = H(A, B) - H(A)
  \]
• Quantum domain: replace $H$ by $S$ and classical random variables by quantum states!

• How to define the quantum conditional entropy?

• There are 2 ways:
  1. A simple way (but which lacks a nice operational interpretation, and can also be negative!!!)

     \[ S(B|A) = S(A, B) - S(A) \]

  2. An operational way, which involves measurements $\{\Pi^a_A\}$ on Alice’s side:

     \[ S(B|\Pi_A) = \sum_a Pr(a) S(\rho^a_B), \quad Pr(a)\rho^a_B = \text{Tr}_A (\Pi^a_A \otimes l_B \rho_{AB} \Pi^a_A \otimes l_B) \]

     and is always positive.
They give rise to 2 (inequivalent) ways of writing the quantum mutual information:

1. \( S(A : B) = S(B) - S(B|A) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \), and
2. \( \chi(A : B, \Pi_A) = S(B) - S(B|\Pi_A) = S(\rho_B) - \sum_a Pr(a)S(\rho^a_B) \), with a choice of \( \{\Pi^a_A\} \) in mind.

The second definition is not symmetric and depends on who’s doing the measurement and also on the measurement operators!

Quantum discord (introduced by H. Ollivier and W. H. Zurek in PRL 88, 017901 (2001)):

\[
D^A(\rho_{AB}, \Pi_A) = S(A : B) - \chi(A : B, \Pi_A).
\]

Sometimes an optimization is made over all possible measurement strategies

\[
D^A(\rho_{AB}) = S(A : B) - \max_{\{\Pi_A\}} \chi(A : B, \Pi_A).
\]
In a sense, discord “removes” the classical correlations from a bipartite quantum state.

\[ D^A(\rho_{AB}) \geq 0 \]

\[ D^A(\rho_{AB}) = 0 \text{ if and only if there exists an orthonormal basis } \{|\psi_k\rangle\} \text{ on Alice’s side such that} \]

\[ \rho_{AB} = \sum_k p_k |\psi_k\rangle \langle \psi_k|_A \otimes \rho_k B, \]


Recently Vedral et al proved in PRL 105, 190502 (2010) that

\[ [L_n, L_m] = 0, \forall n, m, \]

where \( \rho_{AB} = \sum_n c_n L_n \otimes R_n \) is a Schmidt operator form.
There are separable states that have non-zero quantum discord!

Example:

\[ \rho_{AB} = \frac{1}{4} [ |0\rangle \langle 0| \otimes |+\rangle \langle +| + |1\rangle \langle 1| \otimes |-\rangle \langle -| + |+\rangle \langle +| \otimes |1\rangle \langle 1| + |-\rangle \langle -| \otimes |0\rangle \langle 0|]. \]

Hence discord is not an LOCC monotone!

Zero discord environment states is a necessary and sufficient condition for CPTP evolution!
A bilocal implementation $G$ of a gate $U$ on some finite set of separable states $\mathcal{L} = \{\rho_i^{in}\}_{i=1}^N$ (and their convex combinations) is a CPTP map that is implemented by local operations on the subsystems $A$ and $B$, assisted by unlimited classical communication such that for any state $\rho_i^{in} \in \mathcal{L}$

$$G(\rho_i^{in}) = \sum_k K_k \rho_i^{in} K_k^\dagger \equiv U \rho_i^{in} U^\dagger = \rho_i^{out}$$

The dual map $G^\dagger(\rho) := \sum_k K_k^\dagger \rho K_k$ satisfies

$$G^\dagger(\rho_i^{out}) = \rho_i^{in}$$

for all pure input states $\rho_i^{in} \in \mathcal{L}$. Why?

$\rho_i^{out} = G(\rho_i^{in}) = U \rho_i^{in} U^\dagger$ is pure, hence

$$1 = \langle \rho_i^{out} | \rho_i^{out} \rangle = \langle G(\rho_i^{in}) | \rho_i^{out} \rangle = \langle \rho_i^{in} | G^\dagger(\rho_i^{out}) \rangle$$
If the set $\mathcal{L}$ is locally distinguishable, then the gate can be implemented by LOCC (obvious).

If the action creates entanglement, then again the implementation must fail!

However, the absence of entanglement is not sufficient!

If one restricts the local operations to projective measurements and unitaries, then zero discord becomes a necessary criterion for such implementation success.

That’s because local measurements on a state of non-zero discord increases it’s entropy, see PRA 81, 062103 (2010), whereas the gate $U$ does not!
Result 1: If a set $\mathcal{L}$ contains one product state ($|00\rangle$) and the maximally mixed state $(I \otimes I)/4$, and the action of $U$ is realized by LOCC, then all other allowed inputs (and their arbitrary convex combinations) must have zero discord!

Example: the tiles

Figure: Bennett et al states
Result 2: If the set $L$ contains two pure non-orthogonal states, and the unitary operation is such that $D(\rho) \neq D(U\rho U^\dagger)$, where $\rho = p|\psi_1\rangle\langle\psi_1| + (1 - p)|\psi_2\rangle\langle\psi_2|$, for some $0 < p < 1$, then it cannot be implemented on $L$ by LOCC alone.

Conclusions:

- The absence of entanglement in both input and output does not automatically enable a remote implementation by LOCC.
- A discrepancy between local and global information content of separable states (which is captured by the discord) requires entanglement for their processing.
- Entanglement is required for any gate which changes the discord of the states!
- Recent results (arXiv:1006.4402, arXiv:1006.2460) suggest that a change in discord rather than entanglement is the required resource in computational speedup.