Quantum State Estimation

Statistics meets Quantum Mechanics

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Talk Overview

- Introduction to statistical estimation
- Quantum state estimation
- Cube and Tetrahedron measurement schemes and their estimators
- Simulation and analytical results
- Comparison

Quantum State Estimation

Say we have a two-level quantum system

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

where the coefficients are unknown.

Given n copies of the state, our task is to estimate α and β.

Quantum State Estimation

In terms of density operator

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\left(\mathbb{I} + \vec{r}\cdot\vec{\sigma}\right)$$

where $\vec{r}$ is the Bloch vector of length 1.

Next define $\vec{R}$ as our estimate of $\vec{r}$, also of length 1.

$$\vec{R} = (\widehat{X}, \widehat{Y}, \widehat{Z})^T \qquad \vec{R} = (\widehat{r} = 1, \widehat{\theta}, \widehat{\phi})^T$$

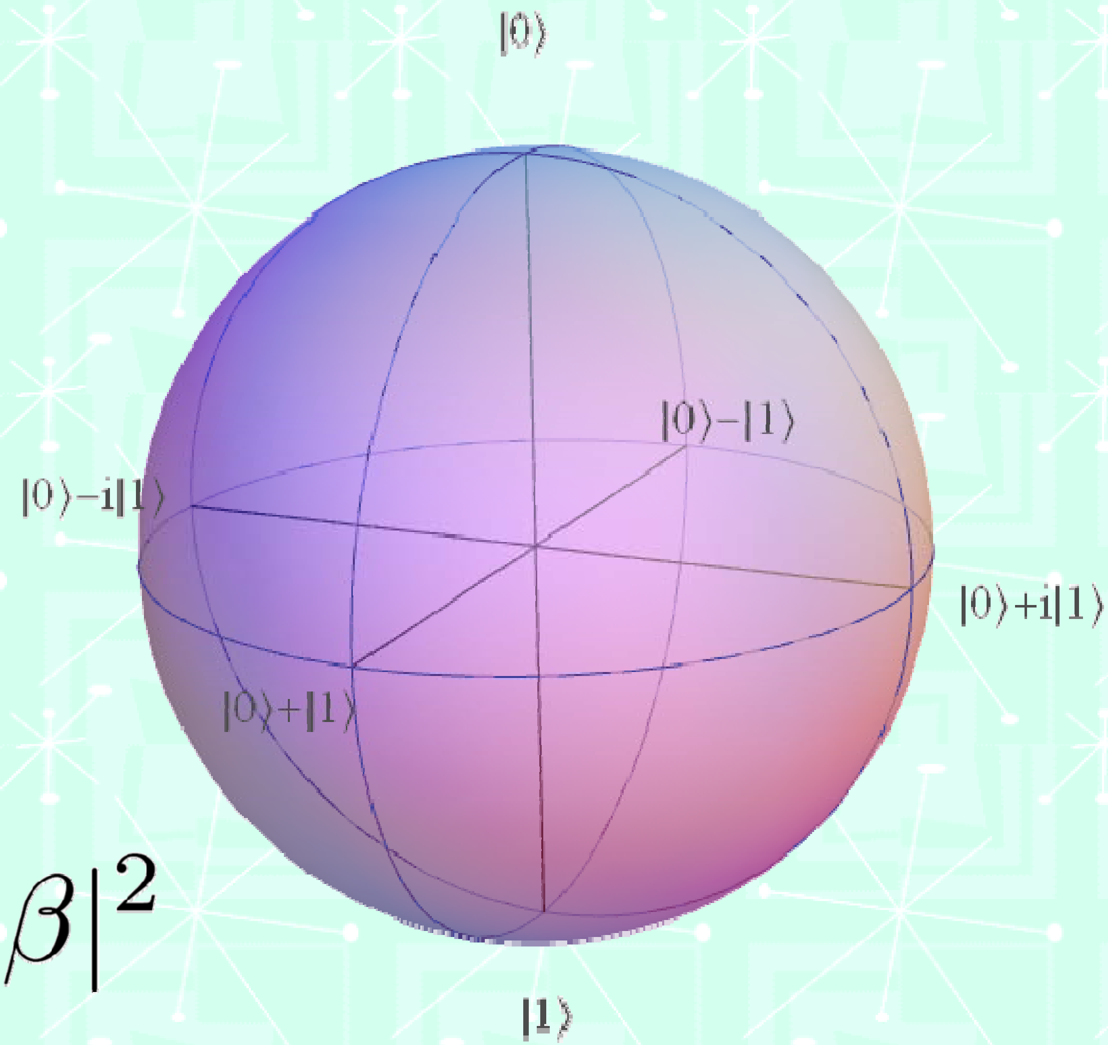
Quantum State Estimation

This talk focuses on

1) Single qubit states
2) Pure states
3) Photonic qubits
4) Non-adaptive measurements
5) One at a time measurements
6) Perfect detectors

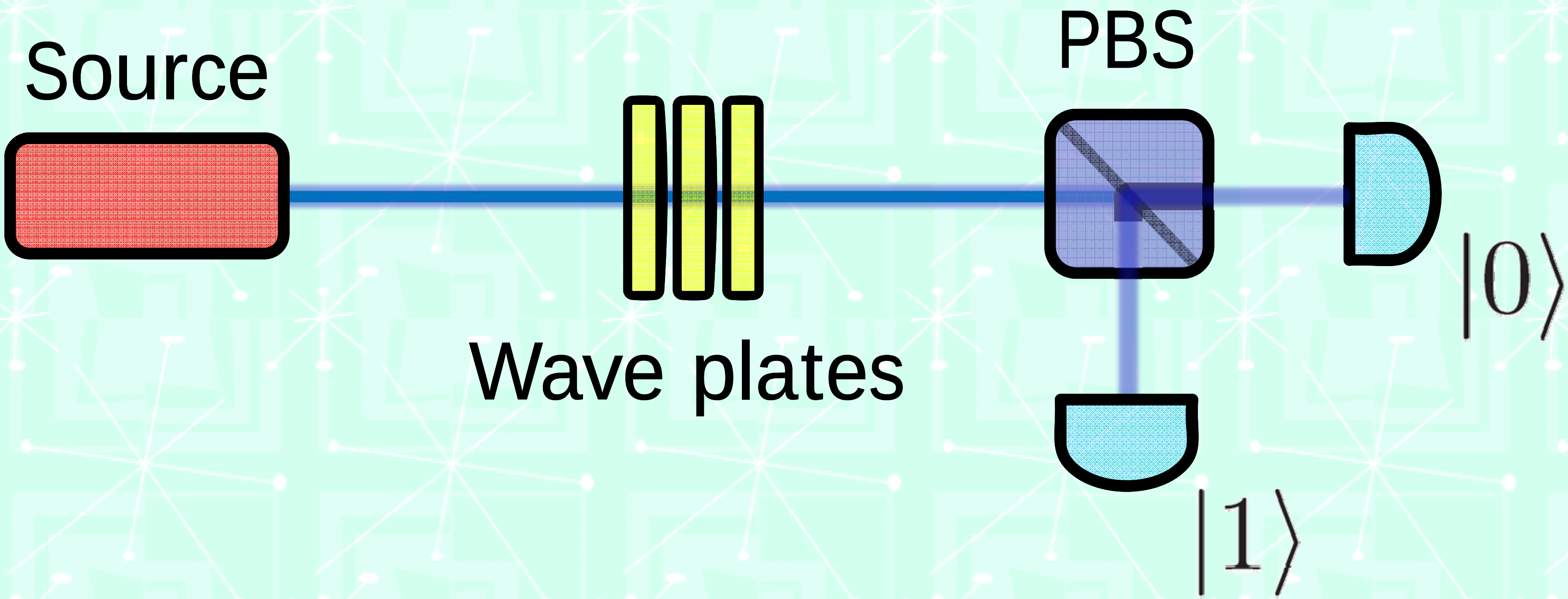
Quantum State Estimation

$$\alpha\,|0\rangle + \beta\,|1\rangle$$



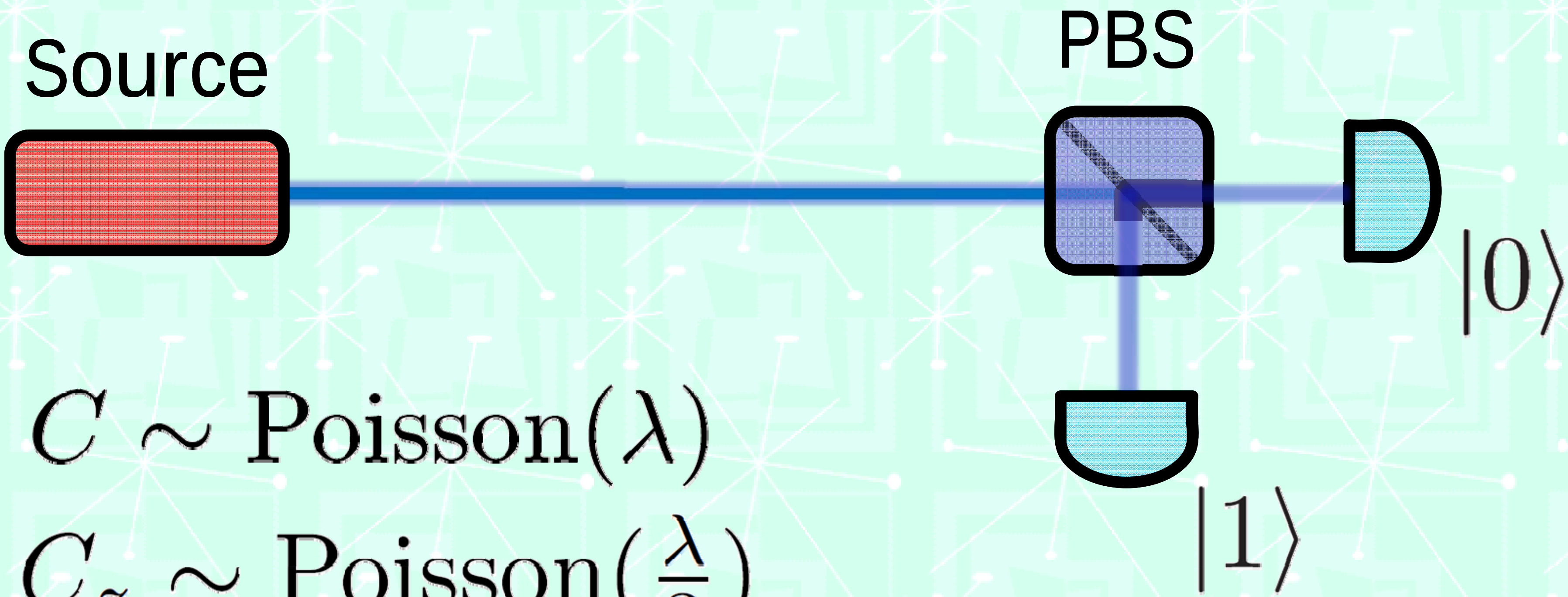
If we only measure in Z basis, we only determine $|\alpha|^2$, $|\beta|^2$

Measurement Scheme



Schematic of Cube scheme

Measurement Scheme



$$C \sim \mathrm{Poisson}(\lambda)$$

$$C_z \sim \mathrm{Poisson}(\tfrac{\lambda}{3})$$

$$C_{+z} \sim \mathrm{Poisson}\left(\tfrac{1}{2}(1+z)\cdot\tfrac{\lambda}{3}\right)$$

$$C_{-z} \sim \mathrm{Poisson}\left(\tfrac{1}{2}(1-z)\cdot\tfrac{\lambda}{3}\right)$$

Measurement Scheme

This suggest the following estimator

$$\widehat{Z}_u := \frac{C_{+z} - C_{-z}}{C_{+z} + C_{-z}}$$

It is an "unbiased" estimator of z.

How to show that $\mathbb{E}(\widehat{Z}_u) = z$? Variance?

Measurement Scheme

Method 1 – Monte Carlo simulation

i) Generate random variates, C_{+z} & C_{-z}

ii) Evaluate $\dfrac{C_{+z} - C_{-z}}{C_{+z} + C_{-z}}$

iii) Repeat i) and ii) 10 million times

iv) Calculate mean and variance of 10 million values

Measurement Scheme

Method 2 – Taylor expansion

i) Taylor expand $\widehat{Z}_u$ about its expectation values

$$\widehat{Z}_u(C_{+z}, C_{-z}) = \widehat{Z}_u(\lambda_{+z}, \lambda_{-z}) + \frac{\partial \widehat{Z}_u}{\partial C_{+z}}(\lambda_{+z}, \lambda_{-z})(C_{+z} - \lambda_{+z}) + \frac{\partial \widehat{Z}_u}{\partial C_{-z}}(\lambda_{+z}, \lambda_{-z})(C_{-z} - \lambda_{-z}) + \cdots$$

ii) Take the expectation value term by term

$$\mathbb{E}(\widehat{Z}_u) = \widehat{Z}_u + \frac{\partial \widehat{Z}_u}{\partial C_{+z}} \mathbb{E}(C_{+z} - \lambda_{+z}) + \frac{\partial \widehat{Z}_u}{\partial C_{-z}} \mathbb{E}(C_{-z} - \lambda_{-z}) + \cdots$$

Measurement Scheme

Method 2 – Taylor expansion

Analytical results

$$\mathbb{E}(\widehat{Z}_u) = z + \mathcal{O}((\tfrac{1}{\lambda})^6)$$

$$\mathbb{V}(\widehat{Z}_u) = (1 - z^2)\left(\frac{3}{\lambda} + \left(\frac{3}{\lambda}\right)^2 + 2\left(\frac{3}{\lambda}\right)^3 + 6\left(\frac{3}{\lambda}\right)^4 + 24\left(\frac{3}{\lambda}\right)^5 + \mathcal{O}((\tfrac{1}{\lambda})^6)\right)$$

Measurement Scheme

So we now have an estimator and we know how well it performs.

$$\widehat{Z}_u := \frac{C_{+z} - C_{-z}}{C_{+z} + C_{-z}}$$

But . . .

Measurement Scheme

Sometimes it is possible to have

$$\widehat{X}_u^2 + \widehat{Y}_u^2 + \widehat{Z}_u^2 > 1$$

So normalize the estimators

$$\widehat{Z} := \frac{\widehat{Z}_u}{\sqrt{\widehat{X}_u^2 + \widehat{Y}_u^2 + \widehat{Z}_u^2}}$$

to give us $\vec{R}_{\mathrm{D}} = (\widehat{X}, \widehat{Y}, \widehat{Z})^T$

Measurement Scheme

There is another estimator

$$\vec{R}_{\mathrm{ML}} = (\widehat{\theta}, \widehat{\phi})^T$$

Estimator has these characteristics

1) Estimator has no closed form expression

2) No analytical results for bias and variance, only simulation

Measurement Scheme

Another measurement scheme is the Tetrahedron scheme.

1) POVM measurement

$$\Pi_i := \frac{1}{4}\left(\mathbb{I} + \vec{d}_i \cdot \vec{\sigma}\right)$$

2) Requires 4 detectors

3) Slightly more complicated, but have been realized experimentally

Measurement Scheme

For the Tetrahedron scheme, we also have two estimators

$$\vec{R}_{\mathrm{D}} = (\widehat{X}, \widehat{Y}, \widehat{Z})^T \qquad \vec{R}_{\mathrm{ML}} = (\widehat{\theta}, \widehat{\phi})^T$$

Measurement Scheme

Let's recap

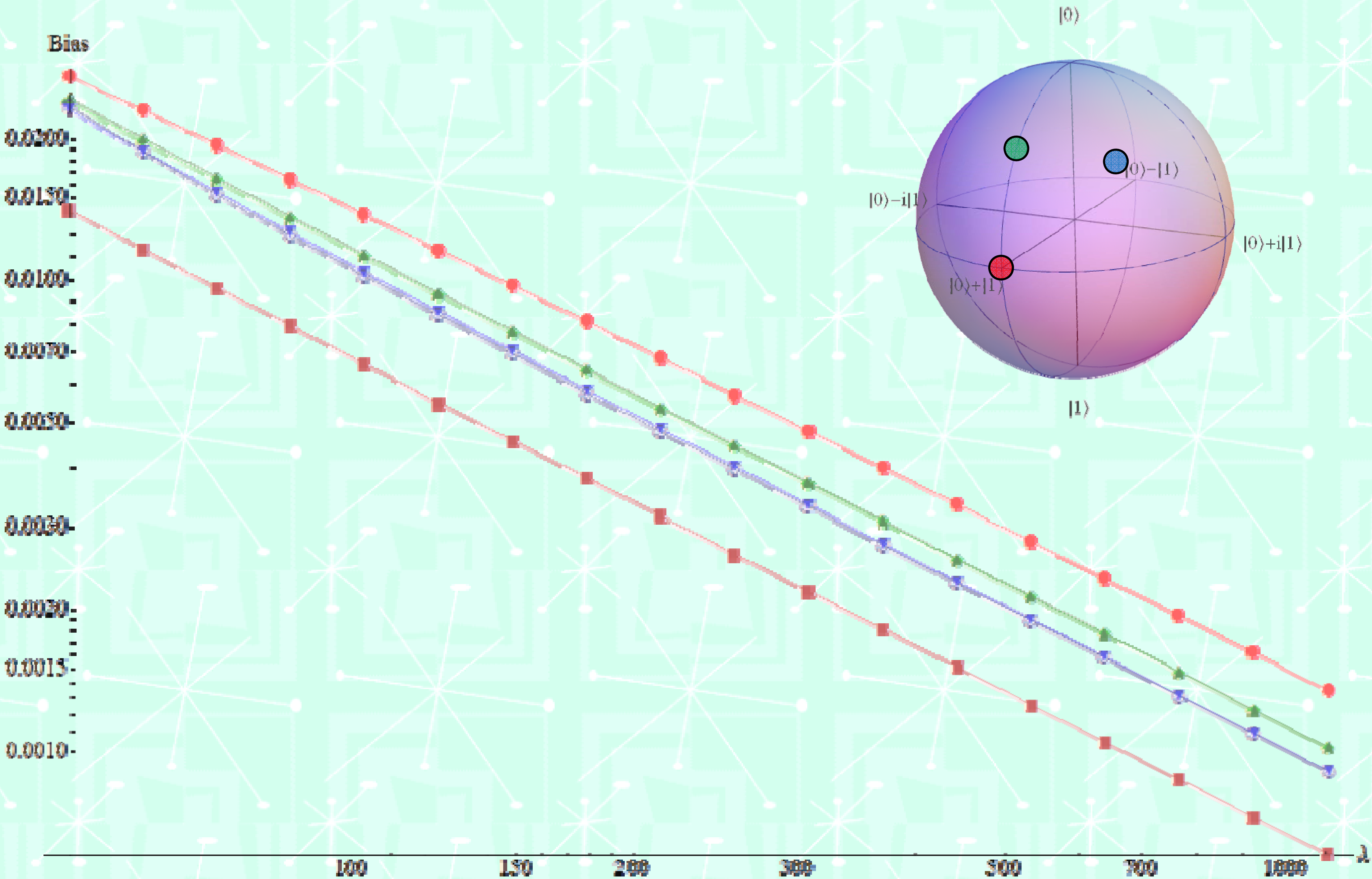
- Two measurement schemes

- Each with two estimators, so total of four

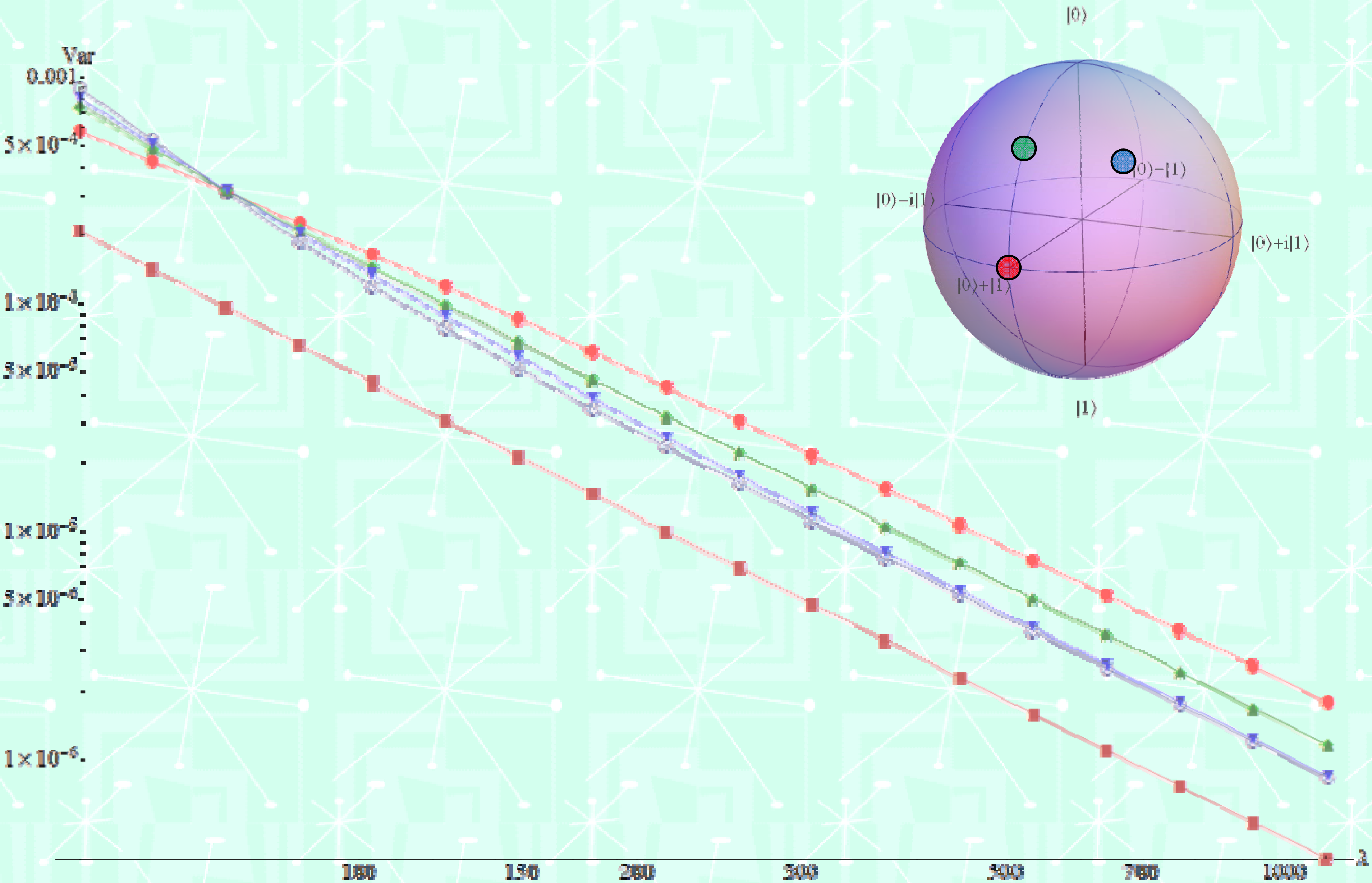
- All the estimators depend on $\vec{r}$

- Have numerical results for all 4 estimators, analytical results only for direct estimators

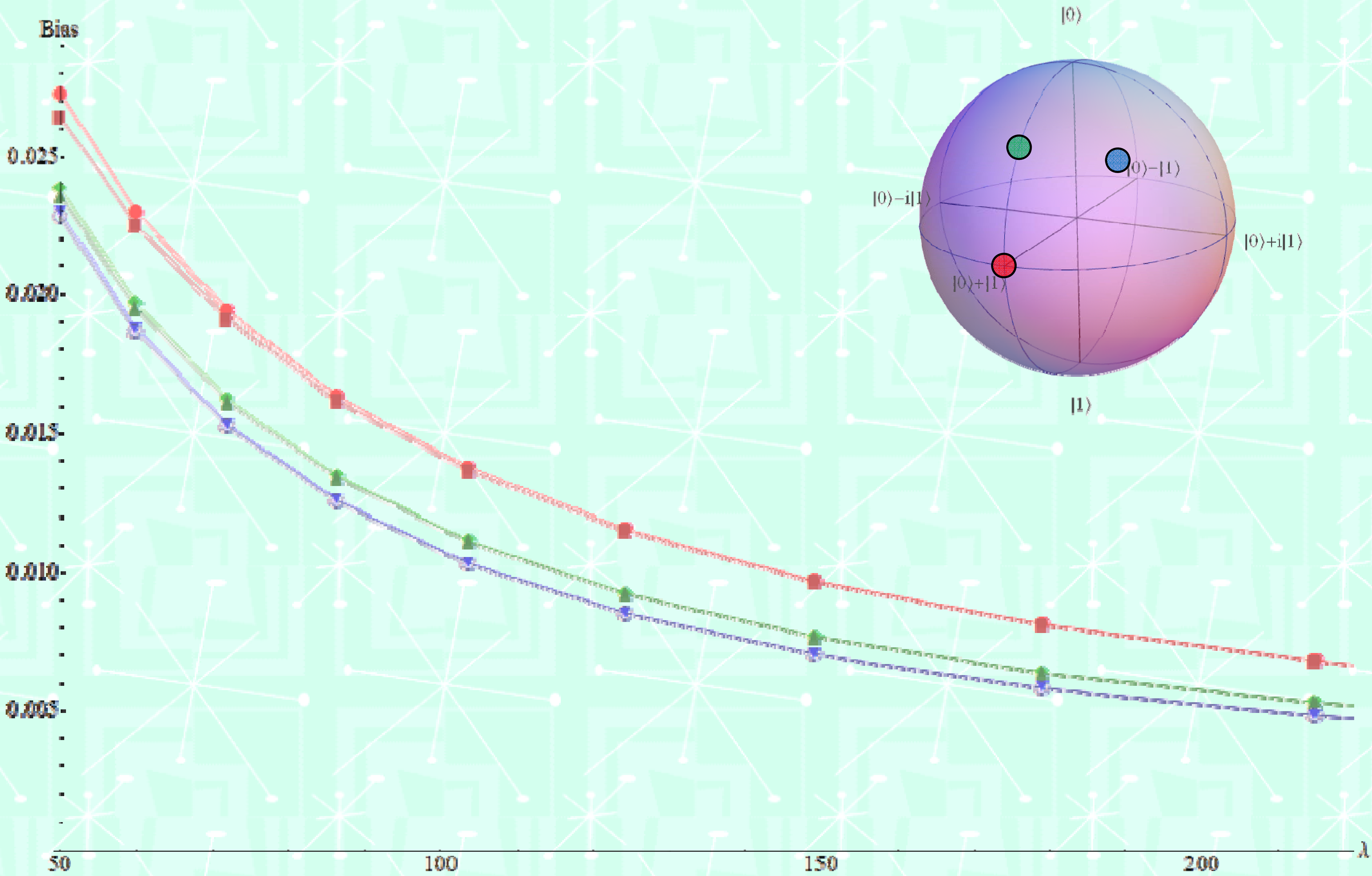
Numerical Results



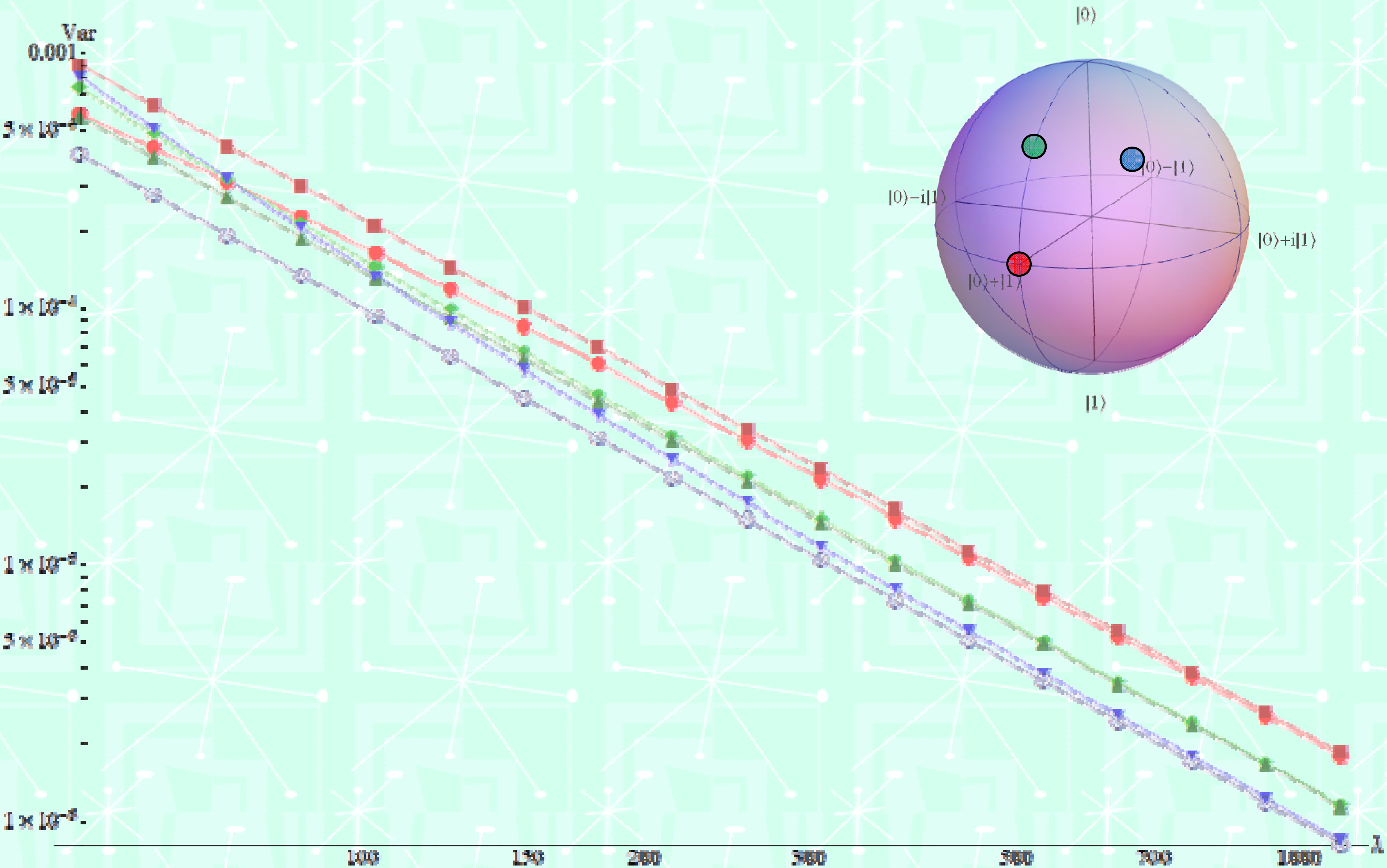
Numerical Results



Numerical Results



Numerical Results



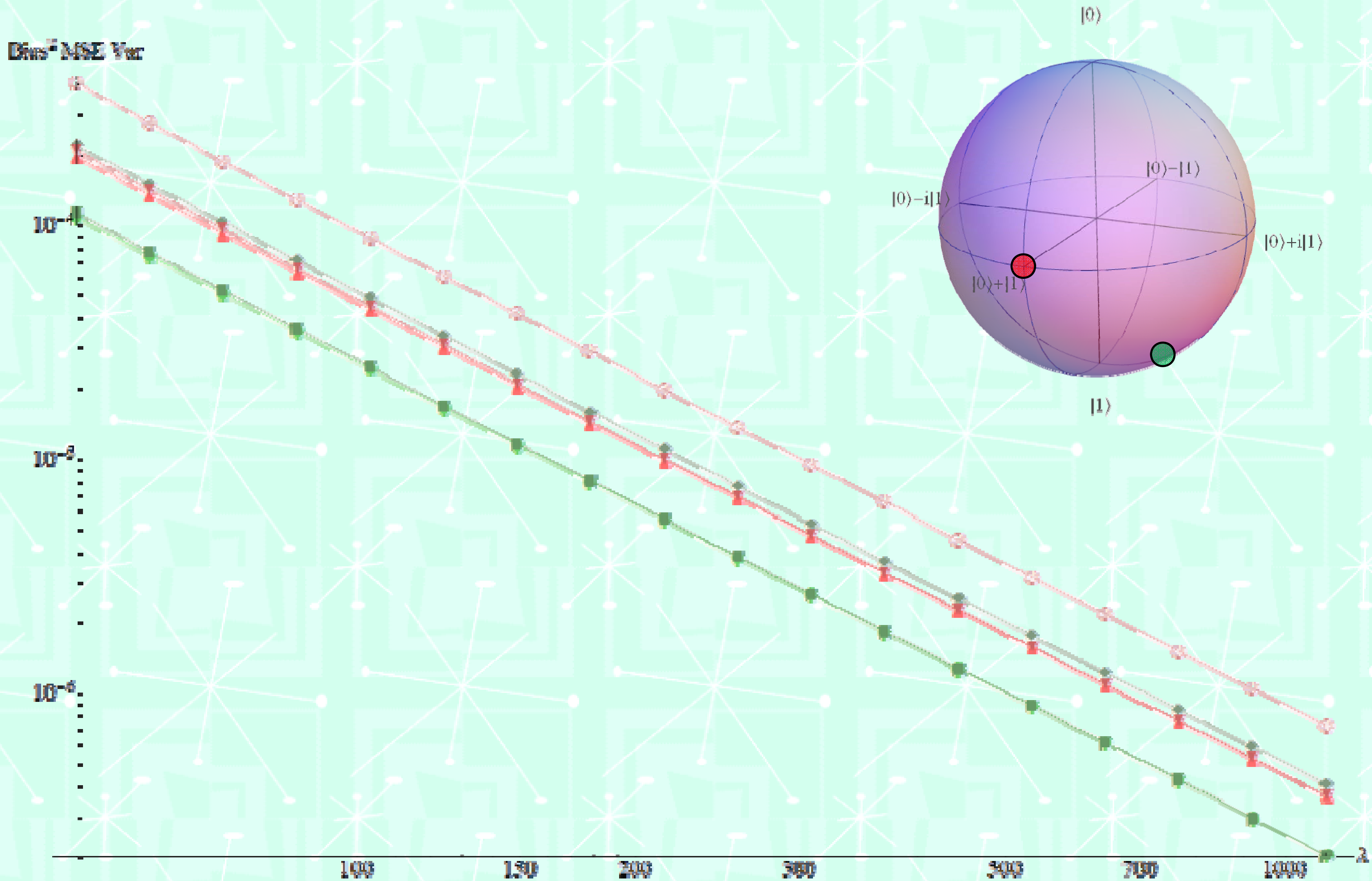
Numerical Results

For analytical results, it is meaningful to talk about best, worst and average case.

Very hard to do the same for numerical results.

But if we consider the symmetries, we can guess where the best/worst cases are.

Numerical Results



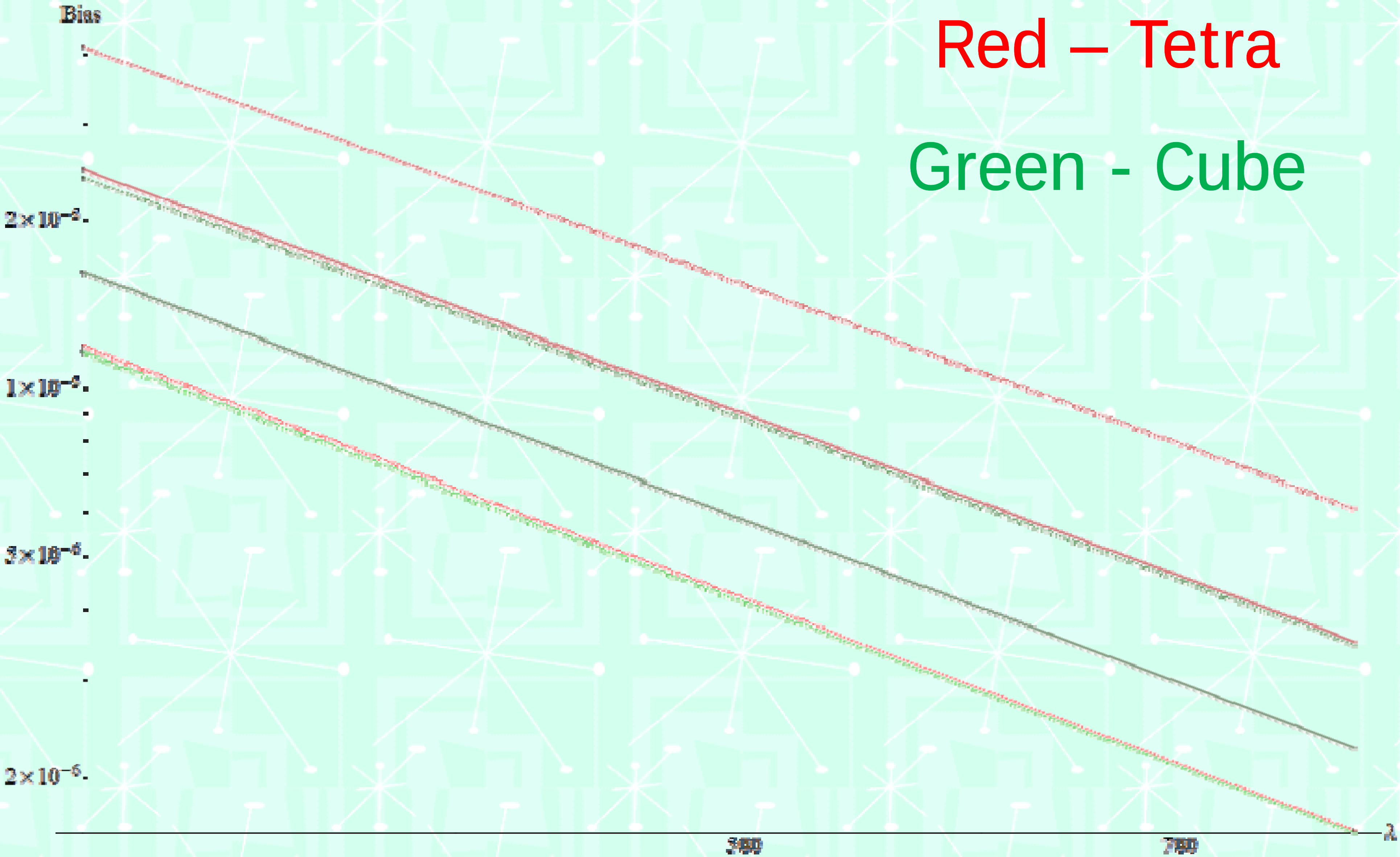
Analytical Results

For analytical results, it is meaningful to talk about best, worst and average case.

But only for direct estimators, not ML estimators.

Numbers for small λ are unreliable.

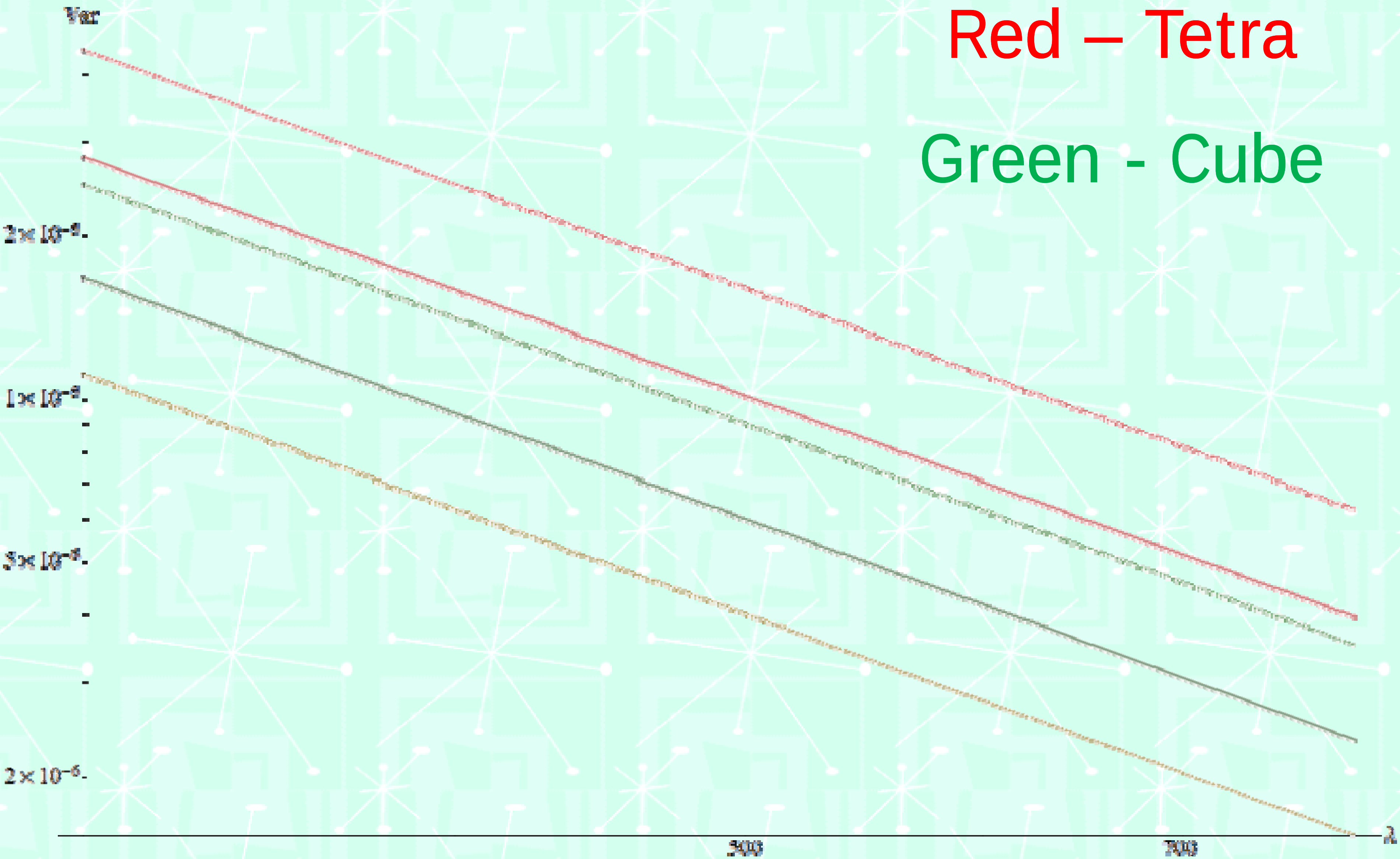
Analytical Results



Red – Tetra

Green - Cube

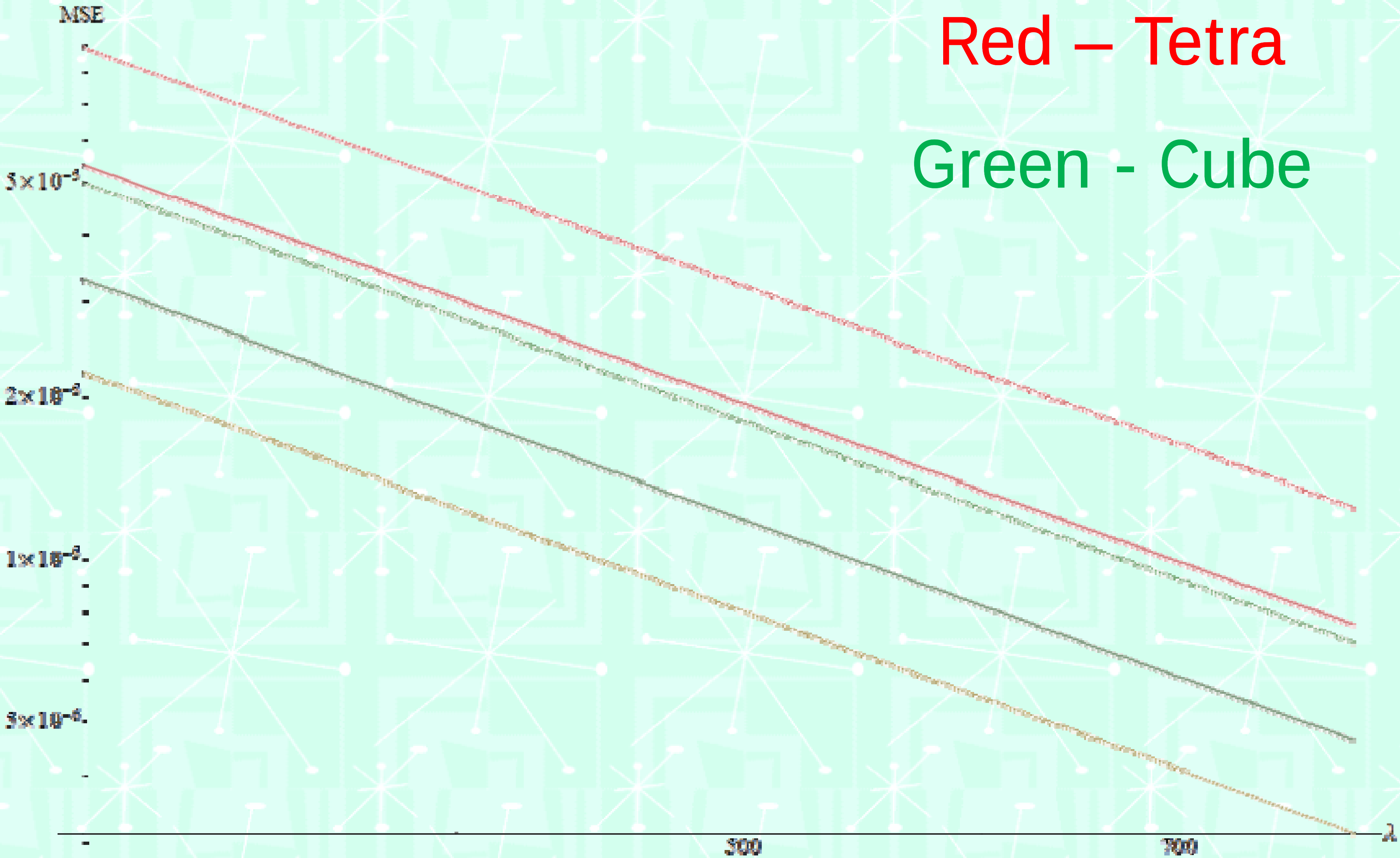
Analytical Results



Red – Tetra

Green - Cube

Analytical Results



Red – Tetra

Green - Cube

Future Research

1) Mixed states

2) Adaptive measurements

3) Two-qubit systems

4) Other estimators??