Operator Quantum Error Correcting Codes

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June 18, 2008

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References

- D.W. Kribbs, R. Laflamme and D. Poulin, Phys. Rev. Lett., **94**, 180501 (2005) - Unified and Generalized Approach to Quantum Error Correction
- D. W. Kribs, R. Laflamme, D. Poulin and M. Lesosky, arXiv:quant-ph/0504189 - Operator Quantum Error Correction
- D. Kribbs, arXiv:math.OA/0506491 - A Brief Introduction to Operator Quantum Error Correction
- M. A. Nielsen and D. Poulin, Phys. Rev. A, **75**, 064304 (2007) - Algebraic and information-theoretic conditions for operator quantum error-correction

The present talk is posted online at http://quantum.phys.cmu.edu under the Quantum Information Seminar heading.

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- Let $\mathcal{E}$ describe the evolution of an open quantum system. Then $\mathcal{E}$ must be positive and trace preserving (maps density operators to density operators).
- This must hold for **any** environment, i.e.

$$Identity_E \otimes \mathcal{E} : \mathcal{L}(\mathcal{H}_E \otimes \mathcal{H}_S) \longrightarrow \mathcal{L}(\mathcal{H}_E \otimes \mathcal{H}_S)$$

must be positive and trace-preserving for all E, or, equivalently, **completely positive trace preserving** (CPTP) map.

- A theorem of Choi and Kraus states that

Kraus representation

Every CPTP map $\mathcal{E}$ has an "operator-sum representation" of the form

$$\mathcal{E}(\rho) = \sum_a E_a \rho E_a^\dagger$$

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- The E_a's are called **noise operators** or **errors** associated with $\mathcal{E}$.
- Shorthand: $\mathcal{E} = \{E_a\}$ when an error model for $\mathcal{E}$ is known.

The standard model of QEC

- Triplet $(\mathcal{R}, \mathcal{E}, \mathcal{C})$, where $\mathcal{C}$ is a subspace, a **quantum code**, of a Hilbert space $\mathcal{H}$, and the **error** $\mathcal{E}$ and **recovery** $\mathcal{R}$ are quantum operations on $\mathcal{L}(\mathcal{H})$.

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- In the trivial case when $\mathcal{E} = \{U\}$ is implemented by a single unitary error operator, the recovery is just the reversal operation $\mathcal{R} = \{U^\dagger\}$

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$$\mathcal{R}(\mathcal{E}(\sigma)) = \sigma, \forall \sigma \in \mathcal{L}(\mathcal{H}).$$

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QEC conditions (Knill-Laflamme)

Given $\mathcal{E} = \{E_a\}$, there exists a recovery operation $\mathcal{R}$ on $\mathcal{C}$ iff there exists a complex matrix $\Lambda = (\lambda_{ab})$ s.t.

$$P_{\mathcal{C}} E_a^{\dagger} E_b P_{\mathcal{C}} = \lambda_{ab} P_{\mathcal{C}}, \forall a, b,$$

where $\mathcal{P}$ is the projector onto the coding space $\mathcal{C}$.

The structure of finite dimensional C^* algebras

- Let $\mathcal{E} = \{E_a\}$ be a quantum operation. Let $\mathcal{A}$ be the C^*-algebra generated by the E_a, i.e. the set of polynomials in the E_a and $E_a^\dagger$. Furthermore,

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- The **noise commutant** associated with $\mathcal{E}$
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$$\mathcal{A}' = \left\{\sigma : [E, \sigma] = 0, \forall E \in \{E_a, E_a^\dagger\}\right\}.$$
- The structure of $\mathcal{A}$ implies that the noise commutant is unitarily equivalent to
$$\mathcal{A}' = \bigoplus_J (\mathbb{I}_{m_J} \otimes \mathcal{M}_{n_J}).$$

Noiseless subsystems for unital channels

- When $\mathcal{E}$ is unital (i.e. $\sum_a E_a E_a^\dagger = \mathbb{I}$), then all states encoded in $\mathcal{A}'$ are immune to the errors of $\mathcal{E}$.

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- Furthermore, it has been shown that when $\mathcal{E}$ is unital, the noise commutant coincides with the set of fixed points for $\mathcal{E}$, i.e.

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- Note that **decoherence-free subspaces** arise as a special case, when $m_J = 1$.

Noiseless subsystems for general channels

- The structure of the algebra $\mathcal{A}$ induces a natural decomposition of the Hilbert space as

$$\mathcal{H} = \bigoplus_J \mathcal{H}_J^A \otimes \mathcal{H}_J^B,$$

where the "noisy subsystems" $\mathcal{H}_J^A$ have dimension m_J and the "noisless subsystems" $\mathcal{H}_J^B$ have dimension n_J.

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- For simplicity, assume that the information is encoded into a single noiseless sector of $\mathcal{L}(\mathcal{H})$, and hence

$$\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K},$$

with dim($\mathcal{H}^A$)=m, dim($\mathcal{H}^B$)=n and dim($\mathcal{K}$)=dim($\mathcal{H}$)-mn.

- Let $\{|\alpha_k\rangle : 1 \leq k \leq m\}$ be an orthonormal basis for $\mathcal{H}^A$ and let

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- Define the semigroup

$$\mathbb{U} = \{\sigma \in \mathcal{L}(\mathcal{H}) : \sigma = \sigma^A \otimes \sigma^B, \text{ for some } \sigma^A \text{ and } \sigma^B\}.$$

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- Let $P_k = P_{kk}$, $P_{\mathbb{U}} = P_1 + P_2 + \ldots + P_m$ and $P_{\mathbb{U}}\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$. Define a map $\mathcal{P}_{\mathbb{U}}(\cdot) = P_{\mathbb{U}}(\cdot)P_{\mathbb{U}}$.

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Equivalent conditions for noiseless subsytems

1. $\forall \sigma^A, \forall \sigma^B, \exists \tau^A : \mathcal{E}(\sigma^A \otimes \sigma^B) = \tau^A \otimes \sigma^B$
2. $\forall \sigma^B, \exists \tau^A : \mathcal{E}(\mathbb{I}^A \otimes \sigma^B) = \tau^A \otimes \sigma^B$
3. $\forall \sigma \in \mathbb{U} : (\mathrm{Tr}_A \circ \mathcal{P}_{\mathbb{U}} \circ \mathcal{E})(\sigma) = \mathrm{Tr}_A(\sigma)$.

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- Note that **decoding** is not necessary.

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3. $\forall \sigma \in \mathbb{U} : (\mathrm{Tr}_A \circ \mathcal{P}_{\mathbb{U}} \circ \mathcal{E})(\sigma) = \mathrm{Tr}_A(\sigma)$.

- Note that **decoding** is not necessary.
- **Operator quantum error correction**: same conditions as for the noiseless subsystems, but decoding (or recovery) is included, i.e.

$$\forall \sigma^A, \sigma^B, \exists \tau^A : \mathcal{R}(\mathcal{E}(\sigma^A \otimes \sigma^B)) = \tau^A \otimes \sigma^B.$$

Necessary and sufficient conditions for OQEC

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Compact algebraic characterization of OQEC

The following conditions are equivalent:

1. $\mathcal{H}^B$ is an $\mathcal{E}$-correcting subsystem with respect to the decomposition $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$.
2. $PE_a^\dagger E_b P = A_{ab} \otimes \mathbb{I}^B$, $\forall a, b$, where P is the projector onto $\mathcal{H}^A \otimes \mathcal{H}^B$ and the A_{ab} are **operators** on A.

Examples

- Let $\{|a\rangle, |b\rangle, |a'\rangle, |b'\rangle\}$ and $\{|a_1\rangle, |b_1\rangle, |a_2\rangle, |b_2\rangle\}$ be two orthonormal bases for $\mathbb{C}^4$. Let P_1 be the projection onto $\text{span}\{|a\rangle, |b\rangle\}$ and P_2 the projection onto $\text{span}\{|a'\rangle, |b'\rangle\}$. Let Q_i, $i = 1, 2$, be the projection onto $\text{span}\{|a_i\rangle, |b_i\rangle\}$. Define operators U_1, U_1', U_2, U_2' on $\mathbb{C}^4$ as $U_1|a\rangle = |a_1\rangle$, $U_1|b\rangle = |b_1\rangle$; $U_1'|a'\rangle = |a_1\rangle$, $U_1'|b'\rangle = |b_1\rangle$; $U_2|a\rangle = |a_2\rangle$, $U_2|b\rangle = |b_2\rangle$; $U_2'|a'\rangle = |a_2\rangle$, $U_2'|b'\rangle = |b_2\rangle$ and put $U_1 P_2 \equiv U_1' P_1 \equiv U_2 P_2 \equiv U_2' P_1 \equiv 0$.

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- These operators are "partial isometries" and satisfy $U_1 = U_1P_1, U_1' = U_1'P_2, U_2 = U_2P_1, U_2' = U_2'P_2$.

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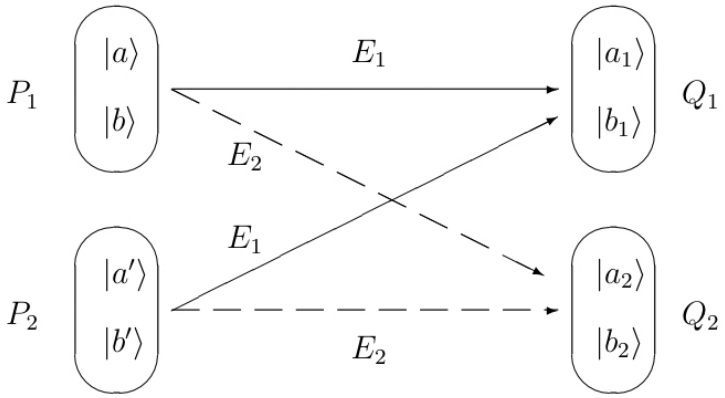
- These operators are "partial isometries" and satisfy $U_1 = U_1P_1, U_1' = U_1'P_2, U_2 = U_2P_1, U_2' = U_2'P_2$.
- The operators $\mathcal{E} = \{E_1, E_2\}$ define a quantum channel where

$$\begin{aligned} E_1 &= \frac{1}{\sqrt{2}}(U_1P_1 + U_1'P_2) \\ E_2 &= \frac{1}{\sqrt{2}}(U_2P_1 - U_2'P_2). \end{aligned}$$

- The action of E_1 and E_2 is indicated below.



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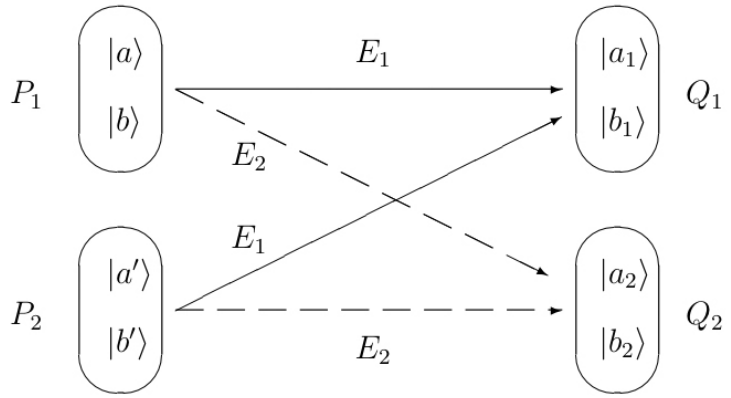

- **Recovery**: define $V_{11} = U_1 P_1$, $V_{12} = U_1' P_2$, $V_{21} = U_2 P_1$, $V_{22} = U_2' P2$ and let the recovery operators be

$$\mathcal{R} = \left\{ \frac{1}{\sqrt{2}} V_{jk}^{\dagger} Q_j : 1 \leq j, k \leq 2 \right\}.$$

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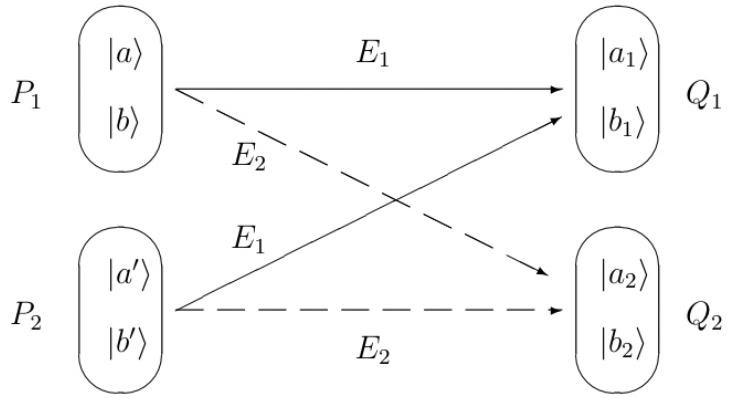

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$$\mathcal{R} = \left\{ \frac{1}{\sqrt{2}} V_{jk}^{\dagger} Q_j : 1 \leq j, k \leq 2 \right\}.$$

- Then all errors induced by $\mathcal{E}$ on $\mathbb{U}_0 \cong \mathbb{I}_2 \otimes \mathcal{M}_2$ can be corrected, i.e.

$$\mathcal{R}(\mathcal{E}(\sigma)) = \sigma, \ \forall \sigma \in \mathcal{L}(\mathbb{C}^4)$$

that have a matrix representation of the form $\sigma = \sigma_1 \oplus \sigma_1$, $\sigma_1 \in \mathcal{M}_2$ with respect to the ordered basis $\{|a\rangle, |b\rangle, |a'\rangle, |b'\rangle\}$.

Discussions

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- QEC is a subset of OQEC (set $\dim(\mathcal{H}^A)=1$).
- As seen in previous examples, there are OQECC that cannot be realized as QECC, so QEC is strictly included in OQEC.
- What about fault-tolerant OQECC?