Operator Quantum Error Correcting Codes

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- D. Kribbs, arXiv:math.OA/0506491 - A Brief Introduction to Operator Quantum Error Correction

The present talk is posted online at http://quantum.phys.cmu.edu under the Quantum Information Seminar heading.
The reversibility postulate of quantum mechanics implies that evolution in a closed quantum system occurs via unitary maps

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An open quantum system is a part of a larger closed one,

\[ \mathcal{H} = \mathcal{H}_E \otimes \mathcal{H}_S \]

where \( \mathcal{H}_E \) is the environment and \( \mathcal{H}_S \) is the open system.
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Let $\mathcal{E}$ describe the evolution of an open quantum system. Then $\mathcal{E}$ must be positive and trace preserving (maps density operators to density operators).
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Let \( \mathcal{E} \) describe the evolution of an open quantum system. Then \( \mathcal{E} \) must be positive and trace preserving (maps density operators to density operators).

This must hold for any environment, i.e.

\[ \text{Identity}_E \otimes \mathcal{E} : \mathcal{L}(\mathcal{H}_E \otimes \mathcal{H}_S) \rightarrow \mathcal{L}(\mathcal{H}_E \otimes \mathcal{H}_S) \]

must be positive and trace-preserving for all \( E \), or, equivalently, completely positive trace preserving (CPTP) map.
A theorem of Choi and Kraus states that

**Kraus representation**

Every CPTP map $\mathcal{E}$ has an "operator-sum representation" of the form

$$\mathcal{E}(\rho) = \sum_a E_a \rho E_a^\dagger$$

for some set of non-unique operators $\{E_a\}$ with

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Shorthand: $\mathcal{E} = \{E_a\}$ when an error model for $\mathcal{E}$ is known.
The standard model of quantum error correction

The standard model of QEC

- Triplet \((\mathcal{R}, \mathcal{E}, \mathcal{C})\), where \(\mathcal{C}\) is a subspace, a quantum code, of a Hilbert space \(\mathcal{H}\), and the error \(\mathcal{E}\) and recovery \(\mathcal{R}\) are quantum operations on \(\mathcal{L}(\mathcal{H})\).
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- In the trivial case when \(\mathcal{E} = \{U\}\) is implemented by a single unitary error operator, the recovery is just the reversal operation \(\mathcal{R} = \{U^\dagger\}\)

\[
\rho \xrightarrow{\mathcal{E}} U\rho U^\dagger \xrightarrow{\mathcal{R}} U^\dagger(U\rho U^\dagger)U = \rho.
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- More generally, the set \((\mathcal{R}, \mathcal{E}, \mathcal{C})\) forms an "error triple" if \(\mathcal{R}\) undoes the effects of \(\mathcal{E}\) on \(\mathcal{C}\).
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\[
\mathcal{R}(\mathcal{E}(\sigma)) = \sigma, \forall \sigma \in \mathcal{L}(\mathcal{H}).
\]
When there exists such an $\mathcal{R}$ for a given pair $\mathcal{E}, \mathcal{C}$, the subspace $\mathcal{C}$ is said to be correctable for $\mathcal{E}$. 
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**QEC conditions (Knill-Laflamme)**

Given $\mathcal{E} = \{E_a\}$, there exists a recovery operation $\mathcal{R}$ on $\mathcal{C}$ iff there exists a complex matrix $\Lambda = (\lambda_{ab})$ s.t.

$$P_C E_a^\dagger E_b P_C = \lambda_{ab} P_C, \forall a, b,$$

where $P$ is the projector onto the coding space $\mathcal{C}$. 
Let $E = \{E_a\}$ be a quantum operation. Let $\mathcal{A}$ be the $C^*$-algebra generated by the $E_a$, i.e. the set of polynomials in the $E_a$ and $E_a^\dagger$. Furthermore,

$$\mathcal{A} \cong \bigoplus_J (\mathcal{M}_{m_J} \otimes \mathbb{I}_{n_J}).$$
Mathematical detour: structure of finite $C^*$ algebras

The structure of finite dimensional $C^*$ algebras

Let $\mathcal{E} = \{E_a\}$ be a quantum operation. Let $\mathcal{A}$ be the $C^*$-algebra generated by the $E_a$, i.e. the set of polynomials in the $E_a$ and $E_a^\dagger$. Furthermore,

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- The **noise commutant** associated with $\mathcal{E}$

$$\mathcal{A}' = \left\{ \sigma : [E, \sigma] = 0, \forall E \in \{E_a, E_a^\dagger\} \right\}.$$
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- The structure of $\mathcal{A}$ implies that the noise commutant is unitarily equivalent to

$$\mathcal{A}' = \bigoplus_j (\mathcal{I}_{m_j} \otimes \mathcal{M}_{n_j}).$$
When $\mathcal{E}$ is unital (i.e. $\sum_a E_a E_a^\dagger = I$), then all states encoded in $\mathcal{A}'$ are immune to the errors of $\mathcal{E}$. Note that decoherence-free subspaces arise as a special case, when $m_J = 1$. 
When $E$ is unital (i.e. $\sum_a E_a E_a^\dagger = I$), then all states encoded in $\mathcal{A}'$ are immune to the errors of $E$.

Furthermore, it has been shown that when $E$ is unital, the noise commutant coincides with the set of fixed points for $E$, i.e.

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Note that **decoherence-free subspaces** arise as a special case, when $m_J = 1$. 
The structure of the algebra $\mathcal{A}$ induces a natural decomposition of the Hilbert space as

$$\mathcal{H} = \bigoplus_J \mathcal{H}^A_J \otimes \mathcal{H}^B_J,$$

where the "noisy subsystems" $\mathcal{H}^A_J$ have dimension $m_J$ and the "noisless subsystems" $\mathcal{H}^B_J$ have dimension $n_J$. 
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For simplicity, assume that the information is encoded into a single noiseless sector of $\mathcal{L}(\mathcal{H})$, and hence

$$\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K},$$

with $\dim(\mathcal{H}^A) = m$, $\dim(\mathcal{H}^B) = n$ and $\dim(\mathcal{K}) = \dim(\mathcal{H}) - mn$. 
Let \( \{ |\alpha_k\rangle : 1 \leq k \leq m \} \) be an orthonormal basis for \( \mathcal{H}^A \) and let
\[
\{ P_{kl} = |\alpha_k\rangle\langle \alpha_l| \otimes I_n : 1 \leq k, k \leq m \}.
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\]

Define the semigroup
\[
\mathbb{U} = \{ \sigma \in \mathcal{L}(\mathcal{H}) : \sigma = \sigma^A \otimes \sigma^B, \text{ for some } \sigma^A \text{ and } \sigma^B \}. 
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Let \( P_k = P_{kk}, \ P_\mathbb{U} = P_1 + P_2 + \ldots + P_m \) and \( P_\mathbb{U} \mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B \).
Define a map \( \mathcal{P}_\mathbb{U}(\cdot) = P_\mathbb{U}(\cdot)P_\mathbb{U} \).
The following three conditions are equivalent for the $\mathcal{H}^B$ sector of the semigroup $\mathbb{U}$ to encode a noiseless subsystem:
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1. $\forall \sigma^A, \forall \sigma^B, \exists \tau^A : \mathcal{E}(\sigma^A \otimes \sigma^B) = \tau^A \otimes \sigma^B$
2. $\forall \sigma^B, \exists \tau^A : \mathcal{E}(\mathbb{I}^A \otimes \sigma^B) = \tau^A \otimes \sigma^B$
3. $\forall \sigma \in \mathbb{U} : (\text{Tr}_A \circ \mathcal{P}_\mathbb{U} \circ \mathcal{E})(\sigma) = \text{Tr}_A(\sigma)$. 

Equivalent conditions for noiseless subsystems

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### Equivalent conditions for noiseless subsystems

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3. $\forall \sigma \in \mathbb{U} : (\text{Tr}_A \circ \mathcal{P}_\mathbb{U} \circ \mathcal{E})(\sigma) = \text{Tr}_A(\sigma)$.

- Note that **decoding** is not necessary.
- **Operator quantum error correction**: same conditions as for the noiseless subsystems, but decoding (or recovery) is included, i.e.

$$\forall \sigma^A, \sigma^B, \exists \tau^A : \mathcal{R}(\mathcal{E}(\sigma^A \otimes \sigma^B)) = \tau^A \otimes \sigma^B.$$
To be used in practical applications, one needs easily testable conditions for a map $\mathcal{E} = \{E_a\}$ to admit a noiseless subsystem.
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Compact algebraic characterization of OQEC

The following conditions are equivalent:

1. $\mathcal{H}^B$ is an $\mathcal{E}$-correcting subsystem with respect to the decomposition $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$.

2. $PE_a^\dagger E_b P = A_{ab} \otimes I^B$, $\forall a, b$, where $P$ is the projector onto $\mathcal{H}^A \otimes \mathcal{H}^B$ and the $A_{ab}$ are operators on $A$. 
Examples

Let \( \{|a\rangle, |b\rangle, |a'\rangle, |b'\rangle\} \) and \( \{|a_1\rangle, |b_1\rangle, |a_2\rangle, |b_2\rangle\} \) be two orthonormal bases for \( \mathbb{C}^4 \). Let \( P_1 \) be the projection onto \( \text{span}\{|a\rangle, |b\rangle\} \) and \( P_2 \) the projection onto \( \text{span}\{|a'\rangle, |b'\rangle\} \). Let \( Q_i, i = 1, 2, \) be the projection onto \( \text{span}\{|a_i\rangle, |b_i\rangle\} \). Define operators \( U_1, U'_1, U_2, U'_2 \) on \( \mathbb{C}^4 \) as

\[
U_1|a\rangle = |a_1\rangle, \quad U_1|b\rangle = |b_1\rangle; \quad U'_1|a'\rangle = |a_1\rangle, \quad U'_1|b'\rangle = |b_1\rangle;
\]

\[
U_2|a\rangle = |a_2\rangle, \quad U_2|b\rangle = |b_2\rangle; \quad U'_2|a'\rangle = |a_2\rangle, \quad U'_2|b'\rangle = |b_2\rangle
\]

and put \( U_1 P_2 \equiv U'_1 P_1 \equiv U_2 P_2 \equiv U'_2 P_1 \equiv 0 \).
Let \{\ket{a}, \ket{b}, \ket{a'}, \ket{b'}\} and \{\ket{a_1}, \ket{b_1}, \ket{a_2}, \ket{b_2}\} be two orthonormal bases for \(\mathbb{C}^4\). Let \(P_1\) be the projection onto \(\text{span}\{\ket{a}, \ket{b}\}\) and \(P_2\) the projection onto \(\text{span}\{\ket{a'}, \ket{b'}\}\). Let \(Q_i, i = 1, 2\), be the projection onto \(\text{span}\{\ket{a_i}, \ket{b_i}\}\). Define operators \(U_1, U'_1, U_2, U'_2\) on \(\mathbb{C}^4\) as

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U_1\ket{a} = \ket{a_1}, \quad U_1\ket{b} = \ket{b_1}; \quad U'_1\ket{a'} = \ket{a_1}, \quad U'_1\ket{b'} = \ket{b_1};
\]
\[
U_2\ket{a} = \ket{a_2}, \quad U_2\ket{b} = \ket{b_2}; \quad U'_2\ket{a'} = \ket{a_2}, \quad U'_2\ket{b'} = \ket{b_2}
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and put \(U_1P_2 \equiv U'_1P_1 \equiv U_2P_2 \equiv U'_2P_1 \equiv 0\).

These operators are ”partial isometries” and satisfy

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U_1 = U_1P_1, \quad U'_1 = U'_1P_2, \quad U_2 = U_2P_1, \quad U'_2 = U'_2P_2.
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Let \( \{|a\rangle, |b\rangle, |a'\rangle, |b'\rangle\} \) and \( \{|a_1\rangle, |b_1\rangle, |a_2\rangle, |b_2\rangle\} \) be two orthonormal bases for \( \mathbb{C}^4 \). Let \( P_1 \) be the projection onto \( \text{span}\{|a\rangle, |b\rangle\} \) and \( P_2 \) the projection onto \( \text{span}\{|a'\rangle, |b'\rangle\} \). Let \( Q_i, i = 1, 2 \), be the projection onto \( \text{span}\{|a_i\rangle, |b_i\rangle\} \). Define operators \( U_1, U'_1, U_2, U'_2 \) on \( \mathbb{C}^4 \) as

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U_1 |a\rangle = |a_1\rangle, \quad U_1 |b\rangle = |b_1\rangle; \quad U'_1 |a'\rangle = |a_1\rangle, \quad U'_1 |b'\rangle = |b_1\rangle;
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U_2 |a\rangle = |a_2\rangle, \quad U_2 |b\rangle = |b_2\rangle; \quad U'_2 |a'\rangle = |a_2\rangle, \quad U'_2 |b'\rangle = |b_2\rangle \]

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These operators are “partial isometries” and satisfy

\[
U_1 = U_1 P_1, \quad U'_1 = U'_1 P_2, \quad U_2 = U_2 P_1, \quad U'_2 = U'_2 P_2.
\]

The operators \( \mathcal{E} = \{E_1, E_2\} \) define a quantum channel where

\[
E_1 = \frac{1}{\sqrt{2}} (U_1 P_1 + U'_1 P_2)
\]

\[
E_2 = \frac{1}{\sqrt{2}} (U_2 P_1 - U'_2 P_2).
\]

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The action of $E_1$ and $E_2$ is indicated below.

$P_1\begin{array}{c} |a\rangle \\
|b\rangle \end{array}$ $\quad E_1 \quad \begin{array}{c} \quad |a_1\rangle \\
\quad |b_1\rangle \end{array}$ $P_2\begin{array}{c} |a'\rangle \\
|b'\rangle \end{array}$ $\quad E_2 \quad \begin{array}{c} \quad |a_2\rangle \\
\quad |b_2\rangle \end{array}$
The action of $E_1$ and $E_2$ is indicated below.

![Diagram](image)

- **Recovery**: define $V_{11} = U_1 P_1$, $V_{12} = U'_1 P_2$, $V_{21} = U_2 P_1$, $V_{22} = U'_2 P2$ and let the recovery operators be

$$\mathcal{R} = \left\{ \frac{1}{\sqrt{2}} V^\dagger_{jk} Q_j : 1 \leq j, k \leq 2 \right\}.$$
The action of $E_1$ and $E_2$ is indicated below.

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$$R = \left\{ \frac{1}{\sqrt{2}} V_{jk}^\dagger Q_j : 1 \leq j, k \leq 2 \right\}.$$

Then all errors induced by $\mathcal{E}$ on $\mathbb{U}_0 \cong \mathbb{I}_2 \otimes \mathcal{M}_2$ can be corrected, i.e.

$$R(\mathcal{E}(\sigma)) = \sigma, \quad \forall \sigma \in \mathcal{L}(\mathbb{C}^4)$$

that have a matrix representation of the form $\sigma = \sigma_1 \oplus \sigma_1$, $\sigma_1 \in \mathcal{M}_2$ with respect to the ordered basis $\{|a\rangle, |b\rangle, |a'\rangle, |b'\rangle\}$. 
QEC is a subset of OQEC (set $\dim(\mathcal{H}^A) = 1$).
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As seen in previous examples, there are OQECC that cannot be realized as QECC, so QEC is strictly included in OQEC.
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What about fault-tolerant OQECC?