Quantum State Estimation

Statistics meets Quantum Mechanics

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Quantum State Estimation

Say we have a two-level quantum system

\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \]

where the coefficients are unknown.

Given \( n \) copies of the state, our task is to estimate \( \alpha \) and \( \beta \).

Want to devise efficient ways to estimate coefficients.
Talk Overview

- Introduction to statistical estimation
- Quantum state estimation
- Cube and Tetrahedron measurement schemes
- Comparison
Statistical Estimation

Say we have a RV with Poisson distribution

\[ X \sim \text{Poisson}(\lambda) \]

\[ \Pr(X = x) = \frac{e^{-\lambda} \lambda^x}{x!} \quad x = 0, 1, 2, \ldots \]

and we want to learn what \( \lambda \) is by sampling values.
Statistical Estimation

We sample \( n \) times from the same distribution, \( \{X_1, X_2, \ldots, X_n\} \).

How to get an estimate of \( \lambda \) from these samples?

We know the expectation value of \( X_i \) is \( \lambda \).

\[
E(X_i) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!} = \lambda
\]
Statistical Estimation

Therefore, one way to estimate lambda is by using the *sample mean* of \( \{X_i\} \).

\[
\hat{\lambda}_1 := \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}
\]

This is also the *Maximum Likelihood* estimator.
Statistical Estimation

We also know the variance of $X$ is $\lambda$, or

$$\mathbb{V}(X_i) \equiv \mathbb{E}(X_i^2) - \mathbb{E}^2(X_i) = \lambda$$

which suggests another estimator

$$\hat{\lambda}_2 := \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$
Statistical Estimation

Are there other estimators?

Which estimator is better?

Three metrics to compare estimators

1) Bias
2) Variance
3) Mean Squared Error, MSE
Bias

Recall the first estimator,

\[ \hat{\lambda}_1 := \frac{1}{n} \sum_{i=1}^{n} X_i \]

Now, \( \hat{\lambda}_1 \) is itself a random variable.

Ideally, \( \hat{\lambda}_1 \) should be distributed around \( \lambda \).

Would like \( \mathbb{E}(\hat{\lambda}_1) \) to be close to \( \lambda \).
Bias

\[ \mathbb{E}(\hat{\lambda}_1) = \sum_{x_1, x_2, \ldots} \left( \frac{1}{n} \sum_i x_i \right) \Pr(x_1) \Pr(x_2) \cdots \]
Bias

\[ E(\hat{\lambda}_1) = E\left( \frac{1}{n} \sum X_i \right) = \frac{1}{n} E\left( \sum X_i \right) \]
\[ = \frac{1}{n} (n \lambda) = \lambda \]

Bias(\hat{\lambda}_1) := E(\hat{\lambda}_1) - \lambda
\[ = 0 \]

\( \hat{\lambda}_1 \) is an unbiased estimator.
Bias

The second estimator,

$$\hat{\lambda}_2 := \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$\text{Bias} (\hat{\lambda}_2) = 0$$
Bias

Let's define another estimator,

\[ \hat{\lambda}_3 := \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 \]

What's the bias of this estimator?
Bias

\[ \text{Bias}(\hat{\lambda}_3) := \mathbb{E}(\hat{\lambda}_3) - \lambda \]

\[
= \mathbb{E}\left(\left( \frac{n-1}{n} \right) \hat{\lambda}_2 \right) - \lambda \\
= \lambda \left( \frac{n-1}{n} \right) - \lambda \\
= -\frac{\lambda}{n}
\]

Estimator is biased, but asymptotically unbiased.
Variance

Recall that both $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are unbiased.

Variance is a measure of the spread.

Therefore, smaller variance is better.
Variance

\[ \mathbb{V}(\hat{\lambda}_1) = \frac{\lambda}{n} \]

\[ \mathbb{V}(\hat{\lambda}_2) = \lambda \frac{(n-1)^2}{n^3} + 2\lambda^2 \frac{(2n-1)}{n^2} \]

\[ \mathbb{V}(\hat{\lambda}_3) = \frac{\lambda}{n} + \frac{2\lambda^2}{n - 1} \]

For \( n > 1 \), \( \mathbb{V}(\hat{\lambda}_1) \leq \mathbb{V}(\hat{\lambda}_3) \leq \mathbb{V}(\hat{\lambda}_2) \) \( \forall \lambda \)
Variance

\[ \mathbb{V}(\hat{\lambda}_1) = \frac{\lambda}{n} \]

\[ \mathbb{V}(\hat{\lambda}_2) = \lambda \frac{(n-1)^2}{n^3} + 2\lambda^2 \frac{(2n-1)}{n^2} \]

Since they are both unbiased, pick the one with smaller variance.

Using Cramer-Rao lower bound, no unbiased estimator has smaller variance than \( \frac{\lambda}{n} \).
Variance

\[ \text{Var} \left( \hat{\lambda}_2 \right) = \lambda \frac{ (n-1)^2 }{ n^3 } + 2\lambda^2 \frac{ (2n-1) }{ n^2 } \]

\[ \text{Var} \left( \hat{\lambda}_3 \right) = \frac{ \lambda }{ n } + \frac{ 2\lambda^2 }{ n - 1 } \]

\( \hat{\lambda}_2 \) has bigger variance.
\( \hat{\lambda}_3 \) has bigger bias.
MSE

\[ \text{MSE}(\hat{\lambda}_2) := \mathbb{V}(\hat{\lambda}_2) + \text{Bias}^2(\hat{\lambda}_2) \]

For \( n > 1 \), \( \text{MSE}(\hat{\lambda}_3) \leq \text{MSE}(\hat{\lambda}_2) \quad \forall \lambda \)

If we have to choose between these two, pick the one with smaller MSE.
Statistical Estimation

Three metrics to compare estimators

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2) Variance

3) Mean Squared Error, MSE
Quantum State Estimation

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Quantum State Estimation

In terms of density operator

\[ \rho = |\psi\rangle \langle \psi| = \frac{1}{2} \left( \mathbb{1} + \vec{r} \cdot \vec{\sigma} \right) \]

where \( \vec{r} \) is the Bloch vector of length 1.

Next define \( \hat{\vec{R}} \) as our estimate of \( \vec{r} \), also of length 1.
Quantum State Estimation

We will be studying the fidelity

\[ \hat{F} := \frac{1}{2} \left( 1 + \vec{r} \cdot \vec{R} \right) \]

We will look at

1) Two measurement schemes
Quantum State Estimation

We will be studying the fidelity

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1) Two measurement schemes
2) Two estimators, $\vec{R}$
Quantum State Estimation

We will be studying the fidelity

\[ \hat{F} := \frac{1}{2} \left( 1 + \vec{r} \cdot \vec{R} \right) \]

We will look at

1) Two measurement schemes
2) Two estimators, \( \vec{R} \)
3) Bias, variance and MSE of \( \hat{F} \) and how it depends on \( \vec{r} \)