Quantum State Estimation

Statistics meets Quantum Mechanics

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Quantum State Estimation

Say we have a two-level quantum system

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

where the coefficients are unknown.

Given n copies of the state, our task is to estimate α and β.

Want to devise efficient ways to estimate coefficients.

Talk Overview

- Introduction to statistical estimation

- Quantum state estimation

- Cube and Tetrahedron measurement schemes

- Comparison

Statistical Estimation

Say we have a RV with Poisson distribution

$$X \sim \text{Poisson}(\lambda)$$

$$\Pr(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad x = 0, 1, 2, \ldots$$

and we want to learn what λ is by sampling values.

Statistical Estimation

We sample n times from the same distribution, $\{X_1, X_2, \ldots, X_n\}$.

How to get an estimate of λ from these samples?

We know the expectation value of X_i is λ.

$$\mathbb{E}(X_i) = \sum_{x=0}^{\infty} x \frac{\mathrm{e}^{-\lambda} \lambda^x}{x!} = \lambda$$

Statistical Estimation

Therefore, one way to estimate lambda is by using the *sample mean* of $\{X_i\}$.

$$\widehat{\lambda}_1 := \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$$

This is also the *Maximum Likelihood* estimator.

Statistical Estimation

We also know the *variance* of X is λ, or

$$\begin{aligned}\mathbb{V}(X_i) &\equiv \mathbb{E}(X_i^2) - \mathbb{E}^2(X_i) \\ &= \lambda\end{aligned}$$

which suggests another estimator

$$\widehat{\lambda}_2 := \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Statistical Estimation

Are there other estimators?

Which estimator is better?

Three metrics to compare estimators

1) Bias
2) Variance
3) Mean Squared Error, MSE

Bias

Recall the first estimator,

$$\widehat{\lambda}_1 := \frac{1}{n}\sum_{i=1}^{n} X_i$$

Now, $\widehat{\lambda}_1$ is itself a random variable.

Ideally, $\widehat{\lambda}_1$ should be distributed around λ.

Would like $\mathbb{E}(\widehat{\lambda}_1)$ to be close to λ.

Bias

$$\mathbb{E}(\widehat{\lambda}_1) = \sum_{x_1, x_2, \ldots} \left(\frac{1}{n} \sum_i x_i \right) \Pr(x_1) \Pr(x_2) \cdots$$

Bias

$$\mathbb{E}(\widehat{\lambda}_1) = \mathbb{E}(\tfrac{1}{n}\sum X_i) = \tfrac{1}{n}\mathbb{E}(\sum X_i)$$
$$= \tfrac{1}{n}(n\lambda) = \lambda$$

$$\mathrm{Bias}(\widehat{\lambda}_1) := \mathbb{E}(\widehat{\lambda}_1) - \lambda$$
$$= 0$$

$\widehat{\lambda}_1$ is an *unbiased* estimator.

Bias

The second estimator,

$$\widehat{\lambda}_2 := \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$\mathrm{Bias}(\widehat{\lambda}_2) = 0$$

Bias

Let's define another estimator,

$$\widehat{\lambda}_3 := \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

What's the bias of this estimator?

Bias

$$\begin{aligned}\mathrm{Bias}(\widehat{\lambda}_3) &:= \mathbb{E}(\widehat{\lambda}_3) - \lambda \\ &= \mathbb{E}\left(\left(\frac{n-1}{n}\right)\widehat{\lambda}_2\right) - \lambda \\ &= \lambda\left(\frac{n-1}{n}\right) - \lambda \\ &= -\frac{\lambda}{n}\end{aligned}$$

Estimator is biased, but asymptotically unbiased.

Variance

Recall that both $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ are unbiased.

Variance is a measure of the spread.

Therefore, smaller variance is better.

Variance

$$\mathbb{V}(\widehat{\lambda}_1) = \frac{\lambda}{n}$$

$$\mathbb{V}(\widehat{\lambda}_2) = \lambda\frac{(n-1)^2}{n^3} + 2\lambda^2\frac{(2n-1)}{n^2}$$

$$\mathbb{V}(\widehat{\lambda}_3) = \frac{\lambda}{n} + \frac{2\lambda^2}{n-1}$$

For $n > 1$, $\mathbb{V}(\widehat{\lambda}_1) \leq \mathbb{V}(\widehat{\lambda}_3) \leq \mathbb{V}(\widehat{\lambda}_2) \;\forall\, \lambda$

Variance

$$\mathbb{V}(\widehat{\lambda}_1) = \frac{\lambda}{n}$$

$$\mathbb{V}(\widehat{\lambda}_2) = \lambda \frac{(n-1)^2}{n^3} + 2\lambda^2 \frac{(2n-1)}{n^2}$$

Since they are both unbiased, pick the one with smaller variance.

Using Cramer-Rao lower bound, no unbiased estimator has smaller variance than $\frac{\lambda}{n}$.

Variance

$$\mathbb{V}(\widehat{\lambda}_2) = \lambda \frac{(n-1)^2}{n^3} + 2\lambda^2 \frac{(2n-1)}{n^2}$$

$$\mathbb{V}(\widehat{\lambda}_3) = \frac{\lambda}{n} + \frac{2\lambda^2}{n-1}$$

$\widehat{\lambda}_2$ has bigger variance.

$\widehat{\lambda}_3$ has bigger bias.

MSE

$$\mathrm{MSE}(\widehat{\lambda}_2) := \mathbb{V}(\widehat{\lambda}_2) + \mathrm{Bias}^2(\widehat{\lambda}_2)$$

For *n* >1, $\mathrm{MSE}(\widehat{\lambda}_3) \le \mathrm{MSE}(\widehat{\lambda}_2) \quad \forall\,\lambda$

If we have to choose between these two, pick the one with smaller MSE.

Statistical Estimation

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Quantum State Estimation

In terms of density operator

$$\rho = |\psi\rangle \langle\psi| = \frac{1}{2}\left(\mathbb{I} + \vec{r} \cdot \vec{\sigma}\right)$$

where $\vec{r}$ is the Bloch vector of length 1.

Next define $\vec{R}$ as our estimate of $\vec{r}$, also of length 1.

Quantum State Estimation

We will be studying the fidelity

$$\widehat{F} := \frac{1}{2}\left(1 + \vec{r} \cdot \vec{R}\right)$$

We will look at

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Quantum State Estimation

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We will look at

1) Two measurement schemes
2) Two estimators, $\vec{R}$
3) Bias, variance and MSE of $\widehat{F}$ and how it depends on $\vec{r}$