CMU Quantum Information Seminar
Graph Structure of Quantum Codes

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1 Cluster States and Graph States

- We can treat the structure of a cluster state as a graph. Hence, the term *graph states* was coined. Briegel and Raussendorf [1, 2, 3] have introduced the computing aspect of the graph states. Schlingemann and Werner [4] try to associate the quantum error-correcting codes with graph states.
- Applying a controlled-phase gate to $|\psi\rangle = |++\rangle$, we obtain

$$|\psi\rangle = |00\rangle + |01\rangle + |10\rangle - |11\rangle = |0+\rangle + |1-\rangle, \tag{1}$$

which becomes maximally entangled. If we have a three-qubit state $|\psi\rangle = |+++\rangle$ initially, we will obtain the cluster state as $|\psi\rangle = |+0+\rangle + |-1-\rangle$, which is also maximally entangled. An important observation is that cluster states are highly entangled states.
- A two-qubit cluster state can be graphically represented by two nodes and a link connecting them, see Figure 1(a). The nodes denote the qubits, and the link means that there is an entanglement (or a controlled-phase gate) between the two qubits. A similar graphical representation goes to three-qubit cluster state, see Figure 1(b).

(a) Two-qubit graph state. (b) Three-qubit graph state.

Figure 1: Line graph representations of the two-qubit and the three-qubit cluster states.

- Similarly, we can construct a more complex cluster state with multiple qubits and with multiple controlled-phase gates among these qubits. In fact, this complex cluster state can be represented by a graph. Hence, we can take the advantage of the graph theory to analyze the cluster state and to discover its properties.

2 Classical Graph Codes

- The graph approach to error-correcting codes was developed in 1960s. Associated to this approach was low-density parity-check code (LDPC code), which was the first code to allow data transmission rates close to the theoretical maximum (i.e., the Shannon Limit). Unfortunately, the LDPC was impractical to implement. The research on LDPC code resumed in 1990s. In 2003, LDPC code beat six turbo codes and became the new standard for the satellite transmission of digital television.

- The other advantages are visualization and scalability. Consider a triple repetition code.

3 Graph Structure of Quantum Codes

- A quantum code is to map the information qudits to another set of qudits. Code parameters: $[[n,k,d]]_q = [[n,k,2e+1]]_{q(k,e)}$, for maximal distance separable (MDS) code. That is, k q-ary qudits are encoded into an n qudit code that is able to correct e errors. A qudit is a q-ary quantum system. When $q = 2$, we call it a qubit.

- The rationale of encoding is to increase entanglement among the encoded qudits. When the code is corrupted by noise, we can utilize entanglement to recover the code, and then further decode the original information qudits. Creating entanglement can be achieved by controlled-phase gates, because these gates produce high entanglement. In other words, cluster states (or graph states) are expected to be useful in the coding design.

- A graph $G = (V, E)$ contains a set $V = \{1, 2, \cdots\}$ of vertices and a set E of edges. Each vertex is a qudit. A subset X of V is the original quantum message we want to encode, and the subset $Y = V \setminus X$ is the encoded message. Adjacency matrix is A. We denote the quantum state of the jth vertex as $|g_j\rangle$ or simply g_j. Note that $g_j \in \mathbb{F}_q$.

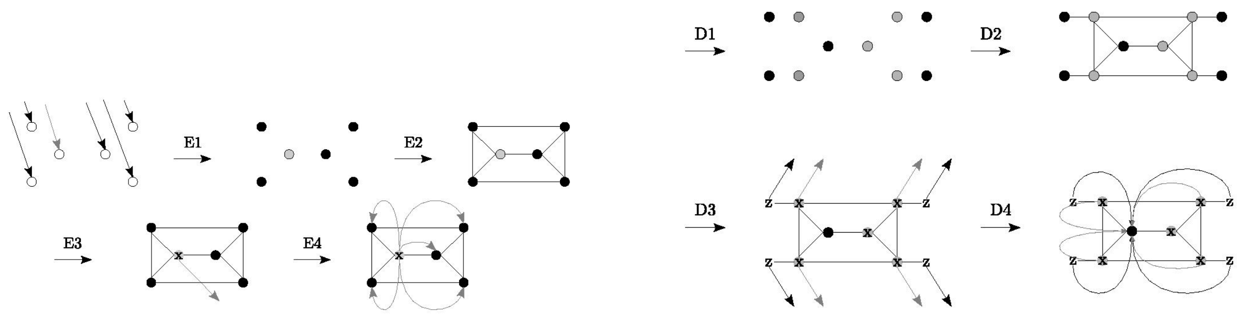
- A subset Z of Y would be corrupted by errors. In other words, the set $\bar{Z} = Y \setminus Z$ of the output vertices is intact.

- Definition: $G^K_L g^L$. Let K and L be the partition of V. $G^K_L g^L := [\sum_{\ell \in L} A_{k\ell} g_\ell]_{k \in K}$. Note that this is a system of $|K|$ equations.

- An error configuration $Z \subset Y$ is detectable by the quantum code C iff the system of equations $G^{\bar{Z}}_{X \cup Z} g^{X \cup Z} = 0$ implies that $g^X = 0$ and $G^X_Z g^Z = 0$.

4 Example: Fivefold Code

- The example of $[[5, 1, 3]]_2$ stabilizer code. See Figure 2 for details.



(a) Encoding (b) Decoding.

Figure 2: Encoding and decoding of a fivefold code [5].

References

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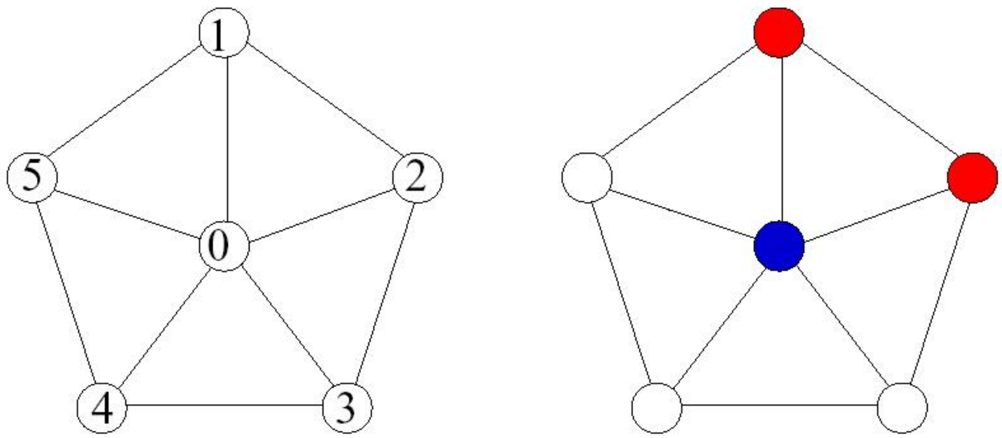
[3] R. Raussendorf and H. J. Briegel, “A one-way quantum computer,” *Phys. Rev. Lett.*, vol. 86, no. 22, pp. 5188–5191, 2001.

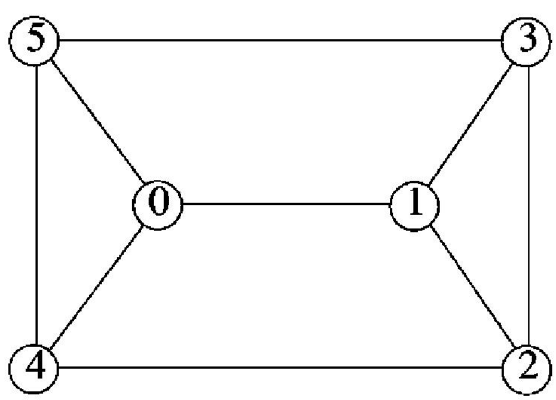
[4] D. M. Schlingemann and R. F. Werner, “Quantum error-correcting codes associated with graphs,” *Phys. Rev. A*, vol. 65, p. 012308, 2001. arXiv: quant-ph/0012111.

[5] D. M. Schlingemann, *Quantum Information Processing with Graph States*. 2005. Habilitation Thesis, Technical University Carolo-Wilhelmina of Braunschweig, Braunschweig, Germany.

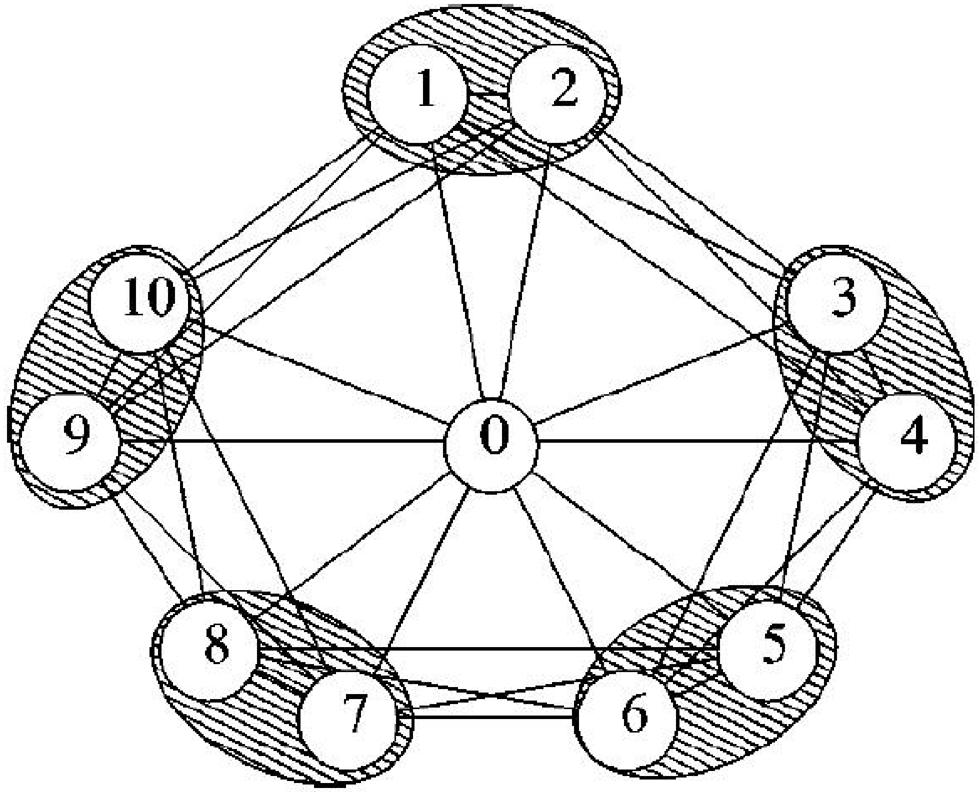


(a) Graph structure. (b) Errors on v_1 and v_2.

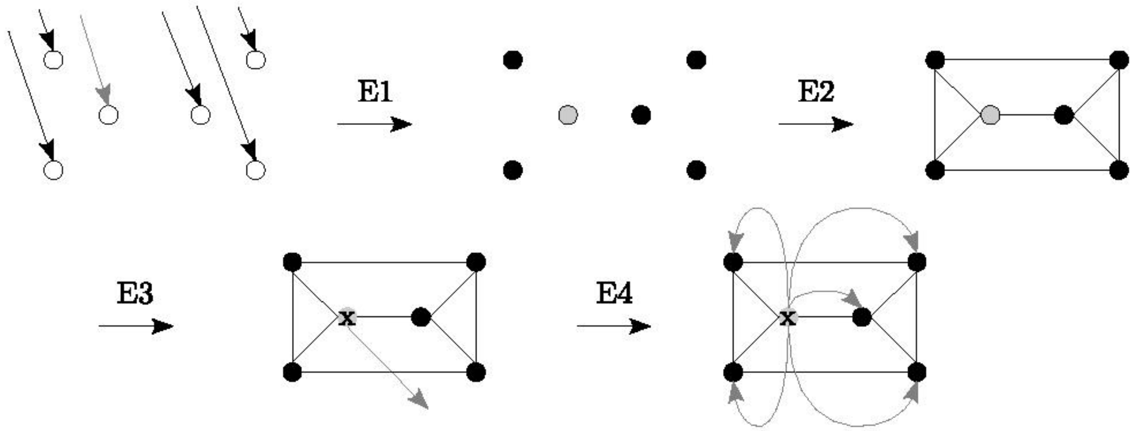
Vertex	Equation
3	$g_0 + g_2 = 0$
4	$g_0 = 0$
5	$g_0 + g_1 = 0$

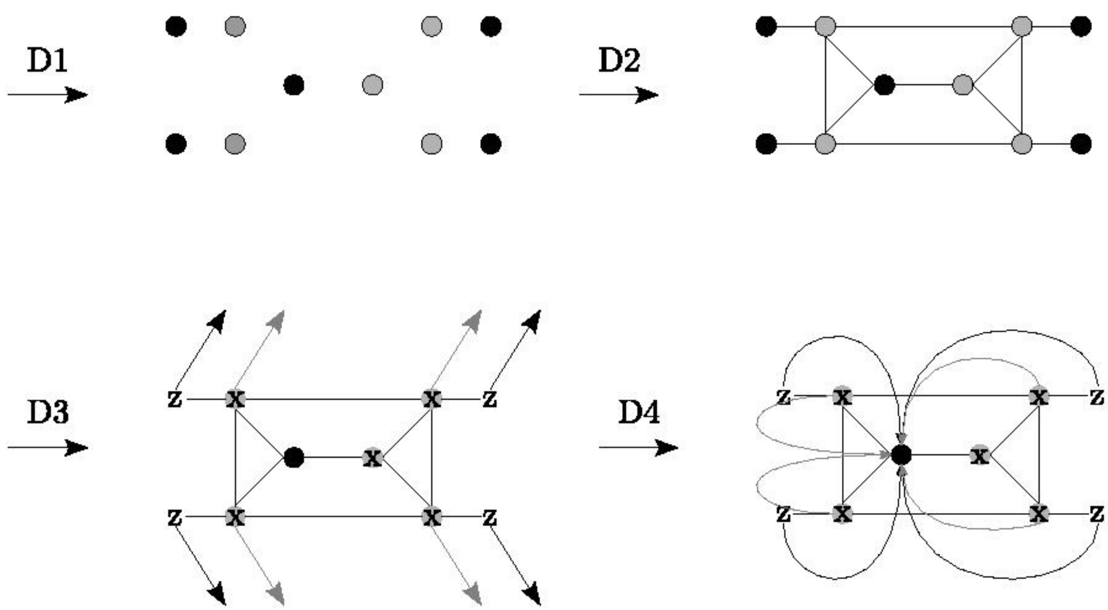


An alternative fivefold code.



A tenfold code obtained from the fivefold code.


E1
E2
E3
E4
x


D1
D2
D3
D4
x
z