Quantum Information Seminar/Workshop

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Abstract

We will discuss the extension of the notion of a graph state from qubits to general d-dimensional qudits. Principal reference: Zhou et al., Phys. Rev. A 68 (2003) 062303.

1 Review of qubit graph states

A **graph** is a set of nodes, which may be joined by vertices. We assume that there are no loops and that the graph is non-directional.

A **graph state** is obtained in the following way: replace the nodes with qubits, initially in the $|+\rangle$ state; apply a controlled-phase gate for each pair of directly joined qubits. In the computational basis the controlled-phase (CP) gate is defined as

$$CP|jk\rangle = (-1)^{jk}|jk\rangle \tag{1}$$

where $j, k \in \{0, 1\}$. The resulting state is called **a graph state**.

2 The qudit case

2.1 Some definitions

Now we consider a d-level quantum system. We would like to generalize the qubit results to this more general case. The ideas are similar. Let $\mathcal{H}$ be the Hilbert space of the system. The computational basis is denoted by $\{|0\rangle, |1\rangle, ..., |d-1\rangle\}$. Now X and Z operators are defined as

$$X|k\rangle = |k \ominus 1\rangle \tag{2}$$

$$Z|k\rangle = \omega^k|k\rangle \tag{3}$$

where $\omega = \exp(2\pi i/3)$.

In the computational basis X and Z are represented as

$$Z = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega^d \end{pmatrix} \tag{4}$$

$$X = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{5}$$

We can define now the Fourier transform of the computational basis

$$|\bar{j}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \omega^{jk} |k\rangle \tag{6}$$

One can easily check that

$$\begin{aligned} X|\bar{j}\rangle &= \omega^{j}|\bar{j}\rangle \\ Z|\bar{j}\rangle &= |\overline{j+1}\rangle \end{aligned} \tag{7}$$

The following commutation relations hold

$$X^j Z^k = \omega^{jk} Z^k X^j \tag{8}$$

By a proper redefinition of X and Z, one can simplify the above commutation relation

$$\overline{XZ} = \omega \overline{ZX}, \tag{9}$$

where

$$\begin{aligned} \overline{Z} &= \omega^{-\frac{d-1}{2} m_1 n_1} Z^{m_1} X^{n_1} \\ \overline{X} &= \omega^{-\frac{d-1}{2} m_2 n_2} Z^{m_2} X^{n_2} \end{aligned} \tag{10}$$

with m_1, n_1, m_2, n_2 chosen such that $(m_1, n_1) = 1$, $(m_2, n_2) = 1$ and $m_1 n_2 - m_2 n_1 = 1$.
$\overline{Z}$ and $\overline{X}$ can simultaneously be written as

$$\overline{Z} = UZU^{\dagger}, \quad \overline{X} = UXU^{\dagger} \tag{11}$$

2.2 The preparation of the graph state

We now restrict to one-dimensional clusters.

1. **Step 1:** Prepare each qudit in the state

$$|+\rangle = \bigotimes_{a=1}^{N} |\bar{0}\rangle_a \tag{12}$$

2. **Step 2:** Apply a generalized controlled-phase gate defined by

$$\begin{aligned} S &= \textstyle\prod_{b \in \mathcal{N}(a)} S_{ab}, \quad \text{with} \\ S_{ab}|j\rangle_a|k\rangle_b &= \omega^{jk}|j\rangle_a|k\rangle_b \end{aligned} \tag{13}$$

   The resulting state $|\Phi\rangle_C = S|+\rangle$ is called the qudit cluster state. It can be shown that $|\Phi\rangle_C$ satisfies the following set of equations

$$X_a^{\dagger} \otimes_{b \in \mathcal{N}(a)} Z_b |\Phi\rangle_C = |\Phi\rangle_C, \quad (\forall) a \tag{14}$$

   Also such set of equations defines a **unique** cluster state.

3 The Clifford group

3.1 Brief review of the Pauli group

For one qubit, the Pauli group consists of the following operators

$$\mathcal{P} = \{\pm I, \pm X, \pm Z, \pm XZ\} \tag{15}$$

In the case of n qubits,

$$\mathcal{P}^n = \mathcal{P} \otimes \mathcal{P} \otimes \ldots \otimes \mathcal{P} \tag{16}$$

The qudit case can be generalized similarly, i.e. the d-dimensional Pauli group for one qudit consists of all d-dimensional operators of the form

$$\omega^l X^j Z^k \tag{17}$$

$j, k, l = 0 \ldots d-1$, $\omega = \exp 2\pi i/d$. The Pauli group for n qudits $\mathcal{P}_d^n$ is defined by the following set of operators

$$\mathcal{P}_d^n = \{\omega^{l_1} X^{j_1} Z^{k_1} \otimes \omega^{l_2} X^{j_2} Z^{k_2} \otimes \ldots \otimes \omega^{l_n} X^{j_n} Z^{k_n}\} \tag{18}$$

The basic commutation relations for the Pauli group can be written as

$$(X^j Z^k)(X^s Z^t) = \omega^{jt-ks}(X^s Z^t)(X^j Z^k) \tag{19}$$

We are interested in the properties of the Pauli group under conjugation by d-dimensional unitary operators.

3.2 The Clifford group

Consider the Pauli group on n qudits, $\mathcal{P}_d^n$. An n-qudit unitary operation $\mathcal{C}$ is defined to be a Clifford operation if

$$\mathcal{C}\mathcal{P}_d^n\mathcal{C}^\dagger = \mathcal{P}_d^n \tag{20}$$

i.e. for every Pauli operation $P \in \mathcal{P}_d^n$, there is another $P^{'} \in \mathcal{P}_d^n$ (which may be different by P) such that $\mathcal{C}P\mathcal{C}^\dagger = P^{'}$.

Hence for every Clifford operation $\mathcal{C}$ we can propagate Pauli operators across $\mathcal{C}$ while $\mathcal{C}$ stays the same (but the Pauli operators generally change). For any n qudits the Clifford operations form a group, called **the Clifford group**.

In the one qubit case ($n = 1$, $d = 2$)

$$HX = ZH, \quad HZ = XH \tag{21}$$

so the Hadamard operator H is a Clifford operation.

Theorem: The Clifford group on n-qudits is generated by Z, H, $P_{\pi/4}$ and $Controlled{-}NOT$ acting in all combinations on any of the qubits (i.e. arbitrary arrays of these gates). Here

$$P_{\pi/4} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \tag{22}$$