Quantum Algorithm for Linear Systems of Equations

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Overview

1) Preliminary Concepts and Classical Algorithms.
2) Overview of Quantum Algorithm
3) Some Details of Quantum Algorithm

References

1) A.W. Harrow, A. Hassidim, S. Lloyd, PRL **103**, 150502 (2009).

2) arXiv:0811.3171v3 [quant-ph].

3) http://www.multimedia.ethz.ch/conferences/2010/qip/?doi=10.3930/ETHZ/AV-37c9a9ce-428f-40c7-964e-6a20c9ec9757

4) http://www.youtube.com/watch?v=KtIPAPyaPOg

Introduction

Let A denote an N by N matrix and **b** be a vector. Then we want to solve for **x**

$$A\,\mathbf{x} = \mathbf{b}$$

This is a very important subroutine in all kinds of calculations.

One class of algorithms to solve for **x** is Gaussian elimination. The complexity is $O(N^3)$.

Introduction

If A is hermitian, positive-definite and sparse, then it is much faster to use iterative methods.

Try to iteratively find **x** that minimizes

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\dagger A\mathbf{x} - \mathbf{b}^\dagger\mathbf{x}$$

A unique global minimum is guaranteed to exist.

Introduction

One of the best algorithm is conjugate gradient method. The complexity is

$$O\left(\kappa \log(1/\epsilon) \cdot N s\right)$$

The algorithm takes $O\left(\kappa \log(1/\epsilon)\right)$ iterations. Each iteration takes $O\left(N s\right)$.

s -- Sparsity of each row.

κ -- Condition number.

ϵ -- Error bound. $\left\|\mathbf{x} - A^{-1}\mathbf{b}\right\|^2 \leqslant \epsilon$.

Overview

The vector **b** is represented as a ket

$$|b\rangle = \sum_{i=1}^{N} b_i |i\rangle$$

The output will be in the form

$$|x\rangle = \sum_{i=1}^{N} x_i |i\rangle$$

Algorithm

Let the eigenvalues and eigenvectors of A be denoted as:

$$A|\mu_j\rangle = \lambda_j|\mu_j\rangle$$

The **b** ket expanded in this eigenbasis is

$$|b\rangle = \sum_{j=1}^{N} \beta_j|\mu_j\rangle$$

Overview

Assuming matrix A is sparse and hermitian, and $1/\kappa \leqslant |\lambda_j| \leqslant 1$

Overview

Assuming matrix A is sparse and hermitian, the complexity of the quantum algorithm is

$$O\left(\frac{\kappa^2}{\epsilon}\log(N)s^2\right)$$

and recall the output is $|x\rangle = \sum_{i=1}^{N} x_i|i\rangle$.

Do you see a problem?

Overview

Assuming matrix A is sparse and hermitian, the complexity of the quantum algorithm is

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and recall the output is $|x\rangle = \sum_{i=1}^{N} x_i|i\rangle$.

It takes at least N steps to readout the vector coefficients.

To overcome this problem, the actual output of algorithm is $\langle x|M|x\rangle$.

Overview

The complexity of the quantum algorithm is

$$O\left(\frac{\kappa^2}{\epsilon}\log(N)s^2\right)$$

Compare this to the classical algorithm,

$$O\left(\kappa\log(1/\epsilon)\cdot Ns\right)$$

The classical complexity is the same whether the output is $\mathbf{x}$ or $\mathbf{x}^\dagger M\mathbf{x}$.

Overview

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$$O\left(\frac{\kappa^2}{\epsilon}\log(N)s^2\right)$$

Compare this to the classical algorithm,

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In some cases the quantum algorithm can be exponentially faster.

Algorithm

Step 1 – Prepare the ancilla

$$|\Psi_0\rangle = \sqrt{\frac{2}{T}} \sum_{\tau=0}^{T-1} \sin \frac{\pi(\tau + \frac{1}{2})}{T} |\tau\rangle$$

Algorithm

Step 1 – Prepare the ancilla and act the following controlled-unitary to it

$$\left(\sum_{\tau=0}^{T-1}[\tau]\otimes \mathrm{e}^{\mathrm{i}A\tau t_0/T}\right)|\Psi_0\rangle \otimes \sum_{j=1}^{N}\beta_j|\mu_j\rangle$$

Ref : D.W. Berry at al, Efficient Quantum Algorithms for Simulating Sparse Hamiltonians, Comm. Math. Phys. **270**, 359-371 (2007).

Algorithm

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produces

$$\sum_{\tau=0}^{T-1}\sum_{j=1}^{N}\sin\frac{\pi(\tau+\frac{1}{2})}{T}|\tau\rangle \otimes \mathrm{e}^{\mathrm{i}\lambda_j\tau t_0/T}\beta_j|\mu_j\rangle$$

Algorithm

Step 2 – Fourier transform the first qudit

$$\sum_{\tau=0}^{T-1}\sum_{j=1}^{N} \sin\frac{\pi(\tau+\frac{1}{2})}{T}|\tau\rangle \otimes \mathrm{e}^{\mathrm{i}\lambda_j \tau t_0/T}\beta_j|\mu_j\rangle$$

given by

$$|\tau\rangle \rightarrow \frac{1}{\sqrt{T}}\sum_{k=0}^{T-1}\mathrm{e}^{2\pi\mathrm{i}\tau k/T}|k\rangle$$

and we get

$$\sum_{k=0}^{T-1}\sum_{j=1}^{N}\alpha_{k|j}\,\beta_j|k\rangle|\mu_j\rangle$$

Algorithm

Step 3 – Add ancilla qubit and perform controlled-rotation

$$\sum_{k=0}^{T-1} \sum_{j=1}^{N} \alpha_{k|j} \; \beta_j |k\rangle |\mu_j\rangle |0\rangle$$

$$\rightarrow \sum_{k=0}^{T-1} \sum_{j=1}^{N} \alpha_{k|j} \; \beta_j |k\rangle |\mu_j\rangle \left(\sqrt{1 - \frac{C^2}{k^2}} |0\rangle + \frac{C}{k} |1\rangle \right)$$

Algorithm

Step 4 – Undo phase estimation and uncompute $|k\rangle$

$$\sum_{k=0}^{T-1} \sum_{j=1}^{N} \alpha_{k|j} \; \beta_j |k\rangle |\mu_j\rangle \left(\sqrt{1 - \frac{C^2}{k^2}} |0\rangle + \frac{C}{k} |1\rangle \right)$$

Algorithm

Step 4 – Undo phase estimation and uncompute $|k\rangle$

$$\sum_{k=0}^{T-1}\sum_{j=1}^{N} \alpha_{k|j}\ \beta_j|\mu_j\rangle \left(\sqrt{1-\frac{C^2}{k^2}}|0\rangle + \frac{C}{k}|1\rangle\right)$$

The coefficients $\alpha_{k|j}$ have the property

$$\alpha_{k|j} = \begin{cases} 1 & \text{if } \lambda_j = 2\pi k/t_0 \\ 0 & \text{otherwise} \end{cases}$$

Algorithm

Step 4 – Undo phase estimation and uncompute $|k\rangle$

$$\sum_{k=0}^{T-1}\sum_{j=1}^{N}\alpha_{k|j}\ \beta_j|\mu_j\rangle\left(\sqrt{1-\frac{C^2}{k^2}}|0\rangle+\frac{C}{k}|1\rangle\right)$$

which simplifies to

$$\sum_{j=1}^{N}\beta_j|\mu_j\rangle\left(\sqrt{1-\frac{C'^2}{\lambda_j^2}}|0\rangle+\frac{C'}{\lambda_j}|1\rangle\right)$$

Algorithm

Step 5 – Measure ancilla qubit

$$\sum_{j=1}^{N} \beta_j |\mu_j\rangle \left(\sqrt{1 - \frac{C'^2}{\lambda_j^2}} |0\rangle + \frac{C'}{\lambda_j} |1\rangle \right)$$

If 1 is measured, we have the state

$$\sum_{j=1}^{N} \lambda_j^{-1} \beta_j |\mu_j\rangle = A^{-1} |b\rangle$$

References

1) A.W. Harrow, A. Hassidim, S. Lloyd, PRL **103**, 150502 (2009).

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