Itô calculus in a nutshell

Vlad Gheorghiu

Department of Physics
Carnegie Mellon University
Pittsburgh, PA 15213, U.S.A.

April 7, 2011

1. Elementary random processes
2. Stochastic calculus
3. Functions of stochastic variables and Itô's Lemma
4. Example: The stock market
5. Derivatives. The Black-Scholes equation and its validity.
6. References

A summary of this talk is available online at
http://quantum.phys.cmu.edu/QIP

Elementary random processes

- Consider a coin-tossing experiment. Head: you win $1, tail: you give me $1.
- Let R_i be the outcome of the i-th toss, $R_i = +1$ or $R_i = -1$ both with probability $1/2$.
- R_i is a random variable.
- $E[R_i] = 0$, $E[R_i^2] = 1$, $E[R_i R_j] = 0$.
- No memory! Same as a fair die, a balanced roulette wheel, but not blackjack!
- Now let $S_i = \sum_{j=1}^{i} R_j$ be the total amount of money you have up to and including the i-th toss.

Random walks

- This is an example of a random walk.

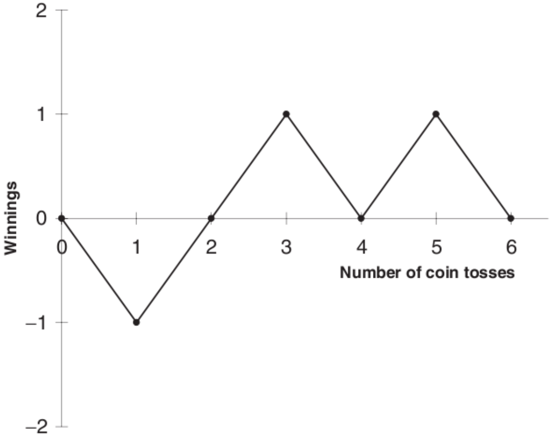

Figure: The outcome of a coin-tossing experiment. From PWQF.

- If we now calculate expectations of S_i *it does matter* what information we have.
- $E[S_i] = 0$ and $E[S_i^2] = E[R_1^2 + 2R_1R_2 + \ldots] = i$.
- The random walk has no memory beyond where it is now. This is the Markov property.
- The random walk has also the martingale property: $E[S_i | S_j, j < i] = S_j$. That is, the conditional expectation of your winnings at any time in the future is just the amount you already hold.
- Quadratic variation: $\sum_{j=1}^{i}(S_j - S_{j-1})^2$. You either win or lose \$1 after each toss, so $|S_j - S_{j-1}| = 1$. Hence the quadratic variation is always i.

Brownian motion

- Now change the rules of the game: allow n tosses in a time t. Second, the size of the bet will not be \$1 but $\$\sqrt{t/n}$.
- Again the Markov and martingale properties are retained and the quadratic variation is still $\sum_{j=1}^{n}(S_j - S_{j-1})^2 = n\left(\sqrt{\frac{t}{n}}\right)^2 = t$.
- In the limit $n \to \infty$ the resulting random walk stays finite. It has an expectation, conditioned on a starting value of zero, of $E[S(t)] = 0$, and a variance $E[S(t)^2] = t$. The limiting process as the time step goes to zero is called Brownian motion, and from now on will be denoted by $X(t)$.

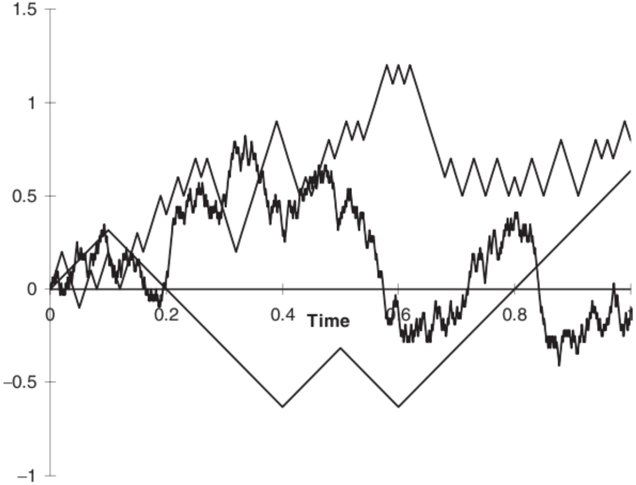

Figure: A series of coin-tossing experiments, the limit of which is a Brownian motion. From PWQF.

Most important properties

- **Continuity:** The paths are continuous, there are no discontinuities. Brownian motion is the continuous-time limit of our discrete time random walk.
- **Markov:** The conditional distribution of $X(t)$ given information up until $\tau < t$ depends only on $X(\tau)$.
- **Martingale:** Given information up until $\tau < t$ the conditional expectation of $X(t)$ is $X(\tau)$.
- **Quadratic variation:** If we divide up the time 0 to t in a partition with $n+1$ partition points $t_j = jt/n$ then

$$\sum_{j=1}^{n} \left(X(t_j) - X(t_{j-1})^2\right)^2 \to t. \quad \text{(Technically "almost surely.")}$$

- **Normality:** Over finite time increments t_{j-1} to t_j, $X(t_j) - X(t_{j-1})$ is Normally distributed with mean zero and variance $t_j - t_{j-1}$.

Stochastic integral

- Let's define the stochastic integral of f with respect to the Browinan motion X by

$$W(t) = \int_0^t f(\tau) dX(\tau) := \lim_{n\to\infty} \sum_{j=1}^{n} f(t_{j-1}) \left(X(t_j) - X(t_{j-1}) \right),$$

  with $t_j = \frac{jt}{n}$.
- The function $f(t)$ which is integrated is evaluated in the summation at the *left-hand point* t_{j-1}, i.e. the integration is **non anticipatory.**. This choice of integration is natural in finance, ensuring that we use no information about the future in our current actions.

Stochastic differential equations

- Stochastic differential equations: The shorthand for a stochastic integral comes from "differentiating" it, i.e.

$$dW = f(t)dX.$$

- For now think of dX as being an increment in X, i.e. a Normal random variable with mean zero and standard deviation $dt^{1/2}$.
- Moving forward, imagine what might be meant by

$$dW = g(t)dt + f(t)dX?$$

It is simply a shorthand for

$$W(t) = \int_0^t g(\tau)d\tau + \int_0^t f(\tau)dX(\tau).$$

The mean square limit

- Examine the quantity $E\left[\left(\sum_{j=1}^{n}\left(X(t_j)-X(t_{j-1})\right)^2-t\right)^2\right]$, where $t_j = jt/n$.
- Because $X(t_j)-X(t_{j-1})$ is Normally distributed with mean zero and variance t/n, i.e. $E\left[\left(X(t_j)-X(t_{j-1})\right)^2\right] = t/n$, one can then easily show that the above expectation behaves like $O(\frac{1}{n})$. As $n \to \infty$ this tends to zero.
- We therefore say

$$\sum_{j=1}^{n}\left(X(t_j)-X(t_{j-1})\right)^2 = t$$

in the "mean square limit".

- This is often written, for obvious reasons, as

$$\int_0^t (dX)^2 = t.$$

Functions of stochastic variables

- If $F = X^2$ is it true that $dF = 2XdX$? NO! The ordinary rules of calculus do not generally hold in a stochastic environment. Then what are the rules of calculus?

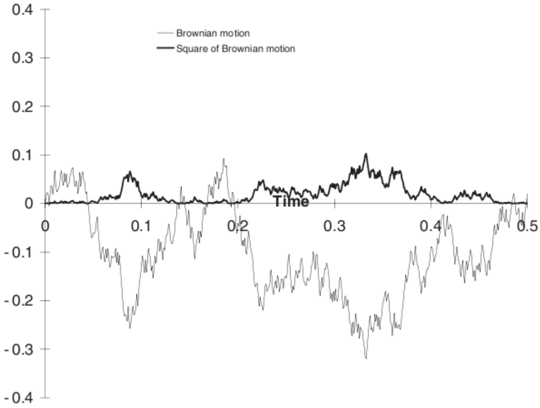

Figure: A realization of a Brownian motion and its square. From PWQF.

Itô's Lemma: A physicist's derivation.

- Let F(X) be an arbitrary function, where $X(t)$ is a Brownian motion. Introduce a very, very small time scale $h = \delta t/n$ so that $F(X(t+h))$ can be approximated by a Taylor series:

$$F(X(t+h)) - F(X(t)) =$$

$$(X(t+h) - X(t))\frac{dF}{dX}(X(t)) + \frac{1}{2}(X(t+h) - X(t))^2\frac{d^2F}{dX^2}(X(t)) + \ldots$$

- From this it follows that

$$(F(X(t+h)) - F(X(t))) + (F(X(t+2h)) - F(X(t+h))) + \ldots$$
$$(F(X(t+nh)) - F(X(t+(n-1)h))) =$$
$$= \sum_{j=1}^{n}(X(t+jh) - X(t+(j-1)h))\frac{dF}{dX}(X(t+(j-1)h))$$
$$+ \frac{1}{2}\frac{d^2F}{dX^2}(X(t))\sum_{j=1}^{n}(X(t+jh) - X(t+(j-1)h))^2 + \ldots$$

- We have used the approximation $\frac{d^2F}{dX^2}(X(t+(j-1)h)) = \frac{d^2F}{dX^2}(X(t))$, consistent with the order we require.
- The first line becomes simply $F(X(t+nh)) - F(X(t)) = F(X(t+\delta t)) - F(X(t))$.
- The second is just the definition of $\int_t^{t+\delta t} \frac{dF}{dX} dX$.
- Finally the last is $\frac{1}{2}\frac{d^2F}{dX^2}(X(t))\delta t$ in the *mean square sense*.
- Thus we have

$$F(X(t+\delta t)) - F(X(t)) = \int_t^{t+\delta t} \frac{dF}{dX}(X(\tau))dX(\tau) + \frac{1}{2}\int_t^{t+\delta t} \frac{d^2F}{dX^2}(X(\tau))d\tau.$$

- Can extend this over longer timescales, from zero up to t, over which F *does* vary substantially, to get

$$F(X(t)) = F(X(0)) + \int_0^t \frac{dF}{dX}(X(\tau))dX(\tau) + \frac{1}{2}\int_0^t \frac{d^2F}{dX^2}(X(\tau))d\tau.$$

- This is the integral version of Itô's Lemma, which is usually written in differential form as

$$dF = \frac{dF}{dX}dX + \frac{1}{2}\frac{d^2F}{dX^2}dt.$$

- Do a naive Taylor series expansion of F, disregarding the nature of X:

$$F(X + dX) = F(X) + \frac{dF}{dX}dX + \frac{1}{2}\frac{d^2F}{dX^2}dX^2.$$

- To get Itô's Lemma, consider that $F(X + dX) - F(X)$ was just the "change in" F and replace dX^2 by dt, remembering $\int_0^t (dX)^2 = t$.
- This is NOT AT ALL rigorous, but has a nice intuitive feeling.

- Coming back to $F = X^2$ and applying Itô's Lemma, we see that F satisfies the stochastic differential equation

$$dF = 2XdX + dt.$$

- In integrated form

$$X^2 = F(X) = F(0) + \int_0^t 2XdX + \int_0^t 1d\tau = \int_0^t 2XdX + t$$

.

- Therefore

$$\int_0^t XdX = \frac{1}{2}X^2 - \frac{1}{2}t.$$

The lognormal random walk

- A stock S is usually modelled as

$$dS = \mu S dt + \sigma S dX,$$

where μ is called the drift and σ the volatility.

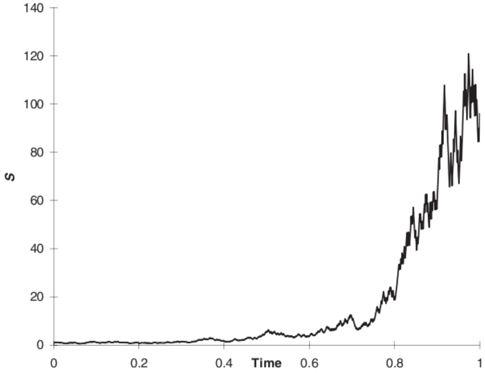

Figure: A realization of $dS = \mu S dt + \sigma S dX$. From PWQF.

- Let $F(S) = \log(S)$ and use Itô's Lemma to get

$$dF = \frac{dF}{dS}dS + \frac{1}{2}\sigma^2 S^2 \frac{d^2F}{dS^2}dt = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dX.$$

- In integrated form,

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma(X(t) - X(0))}.$$

- $S(t)$ is not really a random walk, and is often called a lognormal random walk.

Derivatives

- A derivative (or option) is any function $V(S, t)$ that depends on the underlying stock S.
- The derivative market is HUGE. Are actively traded on most stock exchanges.
- Example: a call option gives you the right (not the obligation) to buy a particular asset for an agreed amount (exercise price, or strike price) at a specified time in the future (expiry or expiration date).
- What is the price of such a contract?
- At expiration, the value is clearly $\max(S - E, 0)$, where E is the strike.
- But what about now? How much would you pay for such an option? The Black-Scholes equation provides the answer.

The Black-Scholes equation

- Consider now a portfolio consisting of a *long position* (we own it) of V and a *short position* (we borrow, owe money) of ΔS assets,

$$\Pi = V(S,t) - \Delta S.$$

- The change in our portfolio from t to $t + dt$ is

$$d\Pi = dV - \Delta dS.$$

- From Itô, one can easily see that V must satisfy

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt.$$

- Hence the portfolio changes by

$$d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt + (\frac{\partial V}{\partial S} - \Delta)dS.$$

- If we choose $\Delta = \frac{\partial V}{\partial S}$, we eliminate the randomness in our portfolio.
- This is called delta hedging. It is a *dynamic hedging* strategy.
- After choosing the quantity Δ as suggested above, we hold a portfolio whose value changes by the amount

$$d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt.$$

- This change is completely *riskless*.
- If we have a completely risk-free change $d\Pi$ in the portfolio value Π then it must be the same (no arbitrage principle) as the growth we would get if we put the equivalent amount of cash in a risk-free interest-bearing account:

$$d\Pi = r\Pi dt.$$

- We then get
$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt = r\Pi dt,$$
from which it follows (remember that $\Pi = V - \Delta S = V - \frac{\partial V}{\partial S} S$)
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$
- This is the famous **Black-Scholes** equation, first written down in 1969, but a few years passed, with Fischer Black and Myron Scholes justifying the model, before it was published. The derivation of the equation was finally published in 1973 and got them a Nobel prize (1997).
- It is a linear parabolic differential equation. Can be reduced to the heat equation.
- Describes the financial instruments under normal conditions. Not valid during market crashes!!!

References

- **Paul Wilmott introduces quantitative finance (PWQF), 2nd ed**, Paul Wilmott, Willey **2007**.
- **Stochastic calculus for finance, vol. I & II**, Steven E. Shreve, Springer Finance textbook **2004**.
- **Probability and Random Processes, 3rd ed**, Geoffrey Grimmett and David Stirzaker, Oxford Univ. Press **2005**.
- Search for Quantitative Finance, Derivatives, Options on Google and Wikipedia :)