Capacities of Quantum Channels

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References

1) Quantum Channel Capacities by Graeme Smith, arXiv:1007.2855

2) QCQI by M. Nielsen and I. Chuang

Overview

1) Various channel capacities
2) Additivity of identical channels
3) Additivity of distinct channels

Definitions

a) Classical capacity

One of the earliest known result is

$$C^{(1)}(\mathcal{N}) = \chi(\mathcal{N})$$

Note the (1) superscript does not denote one-shot capacity.

Definitions

a) Classical capacity

One of the earliest known result is

$$C^{(1)}(\mathcal{N}) = \chi(\mathcal{N})$$

where the Holevo quantity is defined as

$$\chi(\mathcal{N}) := \max_{\{p_x, \rho_x\}} I(X;B)_\sigma$$

$$\sigma := \sum_x p_x |x\rangle\langle x| \otimes \mathcal{N}(\rho_x)$$

Definitions

The full classical capacity can then be characterized as

$$C(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n}\, \chi(\mathcal{N}^{\otimes n})$$

$$= \lim_{n\to\infty} \frac{1}{n}\, C^{(1)}(\mathcal{N}^{\otimes n})$$

This limit cannot actually be computed in general.

Definitions

b) Private classical capacity

It was shown that

$$P^{(1)}(\mathcal{N}) := \max_{\{p_x,\rho_x\}} I(X;B)_\sigma - I(X;E)_\sigma$$

$$\sigma = \sum_x p_x |x\rangle\langle x|_X \otimes U(\rho_x \otimes |0\rangle\langle 0|)U^\dagger$$

Definitions

One can similarly define

$$P(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} P^{(1)}(\mathcal{N}^{\otimes n})$$

where of course

$$P(\mathcal{N}) \geq P^{(1)}(\mathcal{N})$$

Also the private classical capacity cannot be greater than the classical capacity.

Definitions

c) Quantum capacity

It was shown that the coherent information is an achievable rate,

$$Q^{(1)}(\mathcal{N}) := \max_{\{\rho\}} H(B)_\sigma - H(E)_\sigma$$

where $\sigma = U(\rho \otimes |0\rangle\langle 0|_E)U^\dagger$

Definitions

The regularization then gives us

$$Q(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} Q^{(1)}(\mathcal{N}^{\otimes n})$$

and it is obvious that

$$Q(\mathcal{N}) \geq Q^{(1)}(\mathcal{N})$$

Additivity I

An entropic function is additive if

$$f(\mathcal{N}_1 \otimes \mathcal{N}_2) = f(\mathcal{N}_1) + f(\mathcal{N}_2)$$

Additivity I

a) Classical capacity

Recall that

$$C(\mathcal{N}) \geq C^{(1)}(\mathcal{N}) = \chi(\mathcal{N})$$

If we can show that $\chi(\mathcal{N})$ is additive, then we simply have

$$C(\mathcal{N}) = C^{(1)}(\mathcal{N})$$

Additivity I

It was found that $\chi(\mathcal{N})$ is additive for these channels :

Additivity I

It was found that $\chi(\mathcal{N})$ is additive for these channels :

1) Entanglement breaking channels
2) Depolarizing channels
3) Unital qubit channels

$$C(\mathcal{N}) = C^{(1)}(\mathcal{N})$$

This is called a single letter formula.

Additivity I

G. Smith showed that $P(\mathcal{N}) = P^{(1)}(\mathcal{N})$ for degradable channels.

Also, $Q(\mathcal{N}) = Q^{(1)}(\mathcal{N})$ for degradable channels and PPT channels.

For general channels, it is not known if

$$Q(\mathcal{N}) > Q^{(1)}(\mathcal{N})$$

Additivity I

In 2009, Hastings showed there exists a channel satisfying

$$\chi(\mathcal{N} \otimes \mathcal{N}) > 2\chi(\mathcal{N})$$

$$C^{(2)}(\mathcal{N}) > 2C^{(1)}(\mathcal{N})$$

$$\Rightarrow C(\mathcal{N}) > C^{(1)}(\mathcal{N})$$

Bad news : Capacity cannot be computed.

Good news : We can transmit more information using entangled inputs.

Additivity I

By definition the regularized capacity is always additive

$$C(\mathcal{N}^{\otimes n}) = nC(\mathcal{N})$$
$$Q(\mathcal{N}^{\otimes n}) = nQ(\mathcal{N})$$

But what if consider two distinct channels?

Additivity II

In 2008, G. Smith and J. Yard found two channels satisfying

$$Q(\mathcal{N}_1) = 0 \qquad Q(\mathcal{N}_2) = 0$$

But $Q(\mathcal{N}_1 \otimes \mathcal{N}_2) > 0$.

This is a “strong” non-additivity.

Additivity II

Around the same time, several reseachers showed that $P(\mathcal{N})$ is also not additive.

$$P(\mathcal{N}_1) = 0 \qquad P(\mathcal{N}_2) = 0$$

But $P(\mathcal{N}_1 \otimes \mathcal{N}_2) > \frac{1}{8} \log d$.

It is not known if $C(\mathcal{N})$ is additive across different channels or not.

Additivity II

Some comments

- Maybe these quantities are not the relevant measures. Perhaps it cannot be captured with one number.

- However the capacities are still good lower bounds.

- Capacities of composite channels depend on how they are used together.

Outstanding topics

1) Zero-error capacities
2) One-shot capacities

References

1) Quantum Channel Capacities by Graeme Smith, arXiv:1007.2855

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