Degradable Quantum Channels

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References

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- The structure of degradable quantum channels

  Toby S. Cubitt, Mary Beth Ruskai, and Graeme Smith, J. Math. Phys. **49**, 102104 (2008).

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- The quantum capacity with symmetric side channels

  Graeme Smith, John A. Smolin, and Andreas Winter, IEEE Trans. Info. Theory **54**, 9, 4208-4217 (2008).

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- The private classical capacity with a symmetric side channel and its application to quantum cryptography

  Graeme Smith, Phys. Rev. A **78**, 022306 (2008). arXiv:0705.3838v1

Structure of the talk

- Motivations.
- Definition of degradable channels.
- Definition of quantum capacity and private classical capacity.
- Properties of degradable channels.
- Definition of symmetric channels, and their applications.

Motivations

- Useful for studying the quantum capacity of the qubit depolarizing channel (and other channels).
- Possible relation with our study of information in tripartite states / information splitting.
- Possible way of describing decoherence.

Definition of degradable channels

- A channel $\mathcal{N} : \mathcal{H}_A \to \mathcal{H}_B$ can be defined by an isometric embedding $U_\mathcal{N} : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$, followed by a partial trace over the "environment" system E:

$$\mathcal{N}(\rho) = \mathrm{Tr}_E U_\mathcal{N}(\rho). \tag{1}$$

  The *complementary channel* $\mathcal{N}^c : \mathcal{H}_A \to \mathcal{H}_E$ is defined by

$$\mathcal{N}^c(\rho) = \mathrm{Tr}_B U_\mathcal{N}(\rho). \tag{2}$$

  We call a channel $\mathcal{N}$ *degradable* when it may be degraded to its complementary channel $\mathcal{N}^c$, i.e. when there exists a completely positive trace preserving (CPTP) map $\mathcal{T} : \mathcal{H}_B \to \mathcal{H}_E$ such that

$$\mathcal{N}^c = \mathcal{T} \circ \mathcal{N}. \tag{3}$$

- Anti-degradable channel: a channel whose complement is degradable, i.e. there exists a CPTP map $\mathcal{S} : \mathcal{H}_E \to \mathcal{H}_B$ such that $\mathcal{N} = \mathcal{S} \circ \mathcal{N}^c$.

Quantum capacity

- An (n, ϵ) code is defined by an encoding operation $\mathcal{E} : B(\mathcal{H}) \to B(\mathcal{H}_{\mathcal{P}}^{\otimes n})$ and a decoding operation $\mathcal{D} : B(\mathcal{H}_{\mathcal{Q}}^{\otimes n}) \to B(\mathcal{H})$, such that

$$\min_{|\phi\rangle \in \mathcal{H}} F(|\phi\rangle, (\mathcal{D} \circ \mathcal{N}^{\otimes n} \circ \mathcal{E})(|\phi\rangle\langle\phi|)) \geq 1 - \epsilon. \tag{4}$$

  The rate of the code is given by $R = \frac{1}{n} \log \dim \mathcal{H}$. A rate R is called achievable if for all $\epsilon > 0$ and sufficiently large n there exists a code of rate R. The quantum capacity of the channel $Q(\mathcal{N})$ is the supremum of all achievable R. This is also called the subspace transmission capacity.

- The entanglement transmission capacity is denoted by $\tilde{Q}(\mathcal{N})$, which is proven to be equal to $Q(\mathcal{N})$.

Expression for quantum capacity

- Theorem: Given the channel $\mathcal{N}$,

$$Q(\mathcal{N}) = \tilde{Q}(\mathcal{N}) = E(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} \max_{\rho \in \mathcal{H}_{\mathcal{P}}^{\otimes n}} I_c(\rho, \mathcal{N}^{\otimes n}), \qquad (5)$$

where the coherent information I_c is defined as

$$I_c(\rho, \mathcal{N}) = H(\mathcal{N}(\rho)) - H(\mathcal{N}^c(\rho)). \qquad (6)$$

- $Q^{(1)}(\mathcal{N}) \equiv \max_\rho I_c(\rho, \mathcal{N})$ is not additive in general, but is additive for (same or different) degradable channels (arXiv:0705.3838v1, Lemma 4).
- The quantum capacity is not additive, and not convex. (Graeme Smith and Jon Yard, Science **321**, 1812 (2008))

Private classical capacity

- Private classical capacity C_p: regularized capacity of a quantum channel for sending private classical information.
- Secret key capacity K: regularized capacity for generating secret classical correlations between Alice and Bob.
- For the channel $\mathcal{N}$ with output $\mathcal{Q}$ and environment $\mathcal{E}$, the private classical capacity is

$$C_p(\mathcal{N}) = K(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} \max_{X\mathcal{P}} \{I(X;\mathcal{Q}^n) - I(X;\mathcal{E}^n)\}, \quad (7)$$

  where Alice prepends a classical-to-quantum channel $\mathcal{P}|X$ whose output density operators $\rho^{\mathcal{P}}$ are input states for the quantum channel $\mathcal{N}^{\otimes n}$.
- Alternative expressions: arXiv:quant-ph/0304127v6, Eq.(30); Phys. Rev. Lett. **103**, 120501 (2009).

Private classical capacity is not additive

- Private Capacity of Quantum Channels is Not Additive (Ke Li, Andreas Winter, XuBo Zou, GuangCan Guo, Phys. Rev. Lett. **103**, 120501 (2009))
- (Graeme Smith) The one-shot private capacity $C_p^{(1)}$ (same as channel private information) is also not additive, but is additive for degradable channels.
- (Graeme Smith) A channel's (unassisted) private classical capacity may be greater than its quantum capacity, but equality holds for degradable channels.

Properties of degradable channels

- The quantum capacity of a degradable channel is given by the single-letter formula

$$Q(\mathcal{N}) = Q^{(1)}(\mathcal{N}) \equiv \max_{\rho} I_c(\rho, \mathcal{N}), \tag{8}$$

- A degradable channel's private classical capacity is equal to its quantum capacity.
- Bound on Q for convex combinations of degradable channels. (From arXiv:0712.2471v1, Lemma 4) Suppose

$$\mathcal{N} = \sum_i p_i \mathcal{N}_i, \tag{9}$$

where $\mathcal{N}_i$ are degradable. Then $\mathcal{T} = \sum_i p_i \mathcal{N}_i \otimes |i\rangle\langle i|$ is degradable, and

$$Q(\mathcal{N}) \leq \sum_i p_i Q^{(1)}(\mathcal{N}_i). \tag{10}$$

Characterizations of degradable channels

Cubitt *et al*, J. Math. Phys. **49**, 102104 (2008).

- Any channel with simultaneously diagonalizable Kraus operators is degradable.
- The amplitude damping channel is degradable.
- Any qubit channel with exactly two Kraus operators is either degradable or anti-degradable.
- If a degradable channel satisfies that there is a pure input state with full rank output, or satisfies that $d_B = 2$ or 3, then the channel's Choi rank (minimum d_E) is not greater than d_B.
- There exist examples of degradable channels whose Choi rank is greater than d_B.

The symmetric channel

- Suppose $\mathcal{H}_B$ and $\mathcal{H}_E$ are both of dimension d. Let $S = S_d \subset \mathcal{H}_B \otimes \mathcal{H}_E$ be the $d(d+1)/2$-dimensional symmetric subspace between $\mathcal{H}_B$ and $\mathcal{H}_E$. Assume $V_d : \mathcal{C}^{d(d+1)/2} \to S$ is a unitary. We call

$$\mathcal{A}_d(\rho) = \mathrm{Tr}_E V_d \rho V_d^\dagger \tag{11}$$

  the *d-dimensional symmetric channel*. Note that $\mathcal{A}_d$ maps states in a $d(d+1)/2$-dimensional space to (possibly mixed) states on a d-dimensional space. A more general definition of the symmetric channel is simply that $\mathcal{N} = \mathcal{N}^c$.
- The symmetric-side-channel-assisted private classical capacity of a channel $\mathcal{N}$ is simply the private capacity of $\mathcal{N}$ when assisted by an arbitrary symmetric channel.

Capacities of channels assisted by the symmetric side-channel

- The symmetric-side-channel-assisted private classical capacity is additive, convex, and, for degradable channels, equal to the unassisted private classical capacity.
- The symmetric-side-channel-assisted quantum capacity Q_{ss} is an upper bound for the quantum capacity Q. For degradable channels, $Q_{ss} = Q$.

Summary

- Introduced the degradable channels, which include the symmetric channel as a special case.
- Discussed some properties and characterizations of the degradable channels.
- Related the degradable channels to the study of quantum and private classical capacities of channels.