Decoherence Free Subspace & Noiseless Subsystem by example

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References


Overview

1) Decoherence Free Subspace
2) Random Rotation Decoherence Model
3) Noiseless Subsystem
DFS

In QEC, we typically have

\[ \rho \xrightarrow{E} R \xrightarrow{\mathcal{R} \circ \mathcal{E}(\rho)} \]

For a given noise model, we would like to be able to undo the noise.
In QEC, we typically have

\[ \rho \xrightarrow{\mathcal{E}} \mathcal{R} \xrightarrow{\mathcal{R} \circ \mathcal{E}(\rho)} \]

Want to find set of states and decoding for which the following holds

\[ \mathcal{R} \circ \mathcal{E}(\rho) = \rho \]
DFS

DFSs are defined as QEC codes where the decoding operation is not necessary.

\[ \rho \xrightarrow{\mathcal{E}} \mathcal{E}(\rho) \]

We want to find set of states satisfying

\[ \mathcal{E}(\rho) = \rho \]
Example of DFS. Say we have phase noise

\[ |0\rangle \xrightarrow{\mathcal{E}} |0\rangle \quad |1\rangle \xrightarrow{\mathcal{E}} e^{i\phi} |1\rangle \]

Use this two qubit encoding

\[ |0_L\rangle = |01\rangle \quad |1_L\rangle = |10\rangle \]

\[ \alpha |0_L\rangle + \beta |1_L\rangle \xrightarrow{\mathcal{E}} e^{i\phi} (\alpha |0_L\rangle + \beta |1_L\rangle) \]
Random Rotation

Say we have a spin-$\frac{1}{2}$ particle

$$|J = \frac{1}{2}, m\rangle \rightarrow e^{-i\vec{J} \cdot \hat{n}} |J = \frac{1}{2}, m\rangle$$

where $\hat{n}$ is a random vector.

Then can we find a DFS for this noise?
Random Rotation

We know if we have two spin-$\frac{1}{2}$ particles and we look at the singlet state

$$J^2 |s\rangle = 0$$
Random Rotation

We know if we have **two** spin-\(\frac{1}{2} \) particles and we look at the singlet state

\[
J^2 |s\rangle = 0
\]
\[
J_x |s\rangle = 0
\]
\[
J_y |s\rangle = 0
\]
\[
J_z |s\rangle = 0
\]
Random Rotation

We know if we have two spin-$\frac{1}{2}$ particles and we look at the singlet state

\[ J^2 |s\rangle = 0 \]
\[ J_x |s\rangle = 0 \]
\[ J_y |s\rangle = 0 \]
\[ J_z |s\rangle = 0 \]

Therefore \( e^{-i\vec{J} \cdot \hat{n}} |s\rangle = ? \)
Random Rotation

We know if we have two spin-$\frac{1}{2}$ particles and we look at the singlet state

\[ J^2 \ket{s} = 0 \]
\[ J_x \ket{s} = 0 \]
\[ J_y \ket{s} = 0 \]
\[ J_z \ket{s} = 0 \]

Therefore \( e^{-i \vec{J} \cdot \hat{n}} \ket{s} = \ket{s} \) for any \( \hat{n} \)
Random Rotation

The other triplet states $|t_-, t_0, t_+\rangle$ do not have the same property.

The rotation operator will mix them up. But will remain in the triplet subspace.

So the singlet spans a one-dimensional DFS. What is it good for?

How to get a two-dimensional DFS?
Random Rotation

To get a useful DFS we need four spin-$\frac{1}{2}$ particles.

We usually write $|s\rangle = |J = 0, m = 0\rangle$

$|0_L\rangle = |J = 0, m = 0; \lambda = 0\rangle$

$= |s\rangle |s\rangle$

$|1_L\rangle = |J = 0, m = 0; \lambda = 1\rangle$

$= \frac{1}{\sqrt{3}} (|t_+\rangle |t_-\rangle + |t_-\rangle |t_+\rangle$

$- |t_0\rangle |t_0\rangle )$
Random Rotation

Another nicer choice of basis

\[ |0_L\rangle = \frac{1}{\sqrt{6}} \left[ \omega \left( |1001\rangle + |0110\rangle \right) \right. \]
\[ + \omega^2 \left( |0101\rangle + |1010\rangle \right) \]
\[ + |0011\rangle + |1100\rangle \left] \right. \]
\[ \omega = e^{2\pi i / 3} \]

\[ |1_L\rangle = \frac{1}{\sqrt{6}} \left[ \omega^2 \left( |1001\rangle + |0110\rangle \right) \right. \]
\[ + \omega \left( |0101\rangle + |1010\rangle \right) \]
\[ + |0011\rangle + |1100\rangle \left] \right. \]
Random Rotation

Can we construct a DFS with only three spin-$\frac{1}{2}$ particles?

We want to find a two-dimensional subspace such that

\[ e^{-i \vec{J} \cdot \hat{n}} |0_L\rangle = |0_L\rangle \]
\[ e^{-i \vec{J} \cdot \hat{n}} |1_L\rangle = |1_L\rangle \]

It does not exist!
Noiseless Subsystem

Let’s define subsystem first.

If we have $\mathcal{H} = A \oplus B$, then we say $A$ is a subspace.

If we have $\mathcal{H} = (A \otimes B)$, then we say $A$ is a subsystem.

If we have $\mathcal{H} = (A \otimes B) \oplus \mathcal{K}$, then we say $A$ is a subsystem.

All subspaces are subsystems.
Noiseless Subsystem

Decoherence free subsystem = noiseless subsystem

Recall for DFS we have $\mathcal{E}(\rho) = \rho$. For NS we have

$\mathcal{E}(\rho \otimes \sigma) = \rho \otimes \sigma'$
Noiseless Subsystem

Claim: There is a noiseless subsystem in three spin-$\frac{1}{2}$ particles.

Consider the irreps of rotation group.

\[ 2 \otimes 2 \cong 2 \otimes 2 \cong 1 \oplus 3 \]
Noiseless Subsystem

Claim: There is a noiseless subsystem in three spin-$\frac{1}{2}$ particles.

Consider the irreps of rotation group.

\[
2 \otimes 2 \simeq 2 \otimes 2 \simeq 1 \oplus 3 \\
2 \otimes 3 \simeq 2 \otimes 2 \otimes 2 \simeq (1 \oplus 3) \otimes 2
\]
Noiseless Subsystem

Claim: There is a noiseless subsystem in three spin-$\frac{1}{2}$ particles.

Consider the irreps of rotation group.

\[
2 \otimes^2 \cong 2 \otimes 2 \cong 1 \oplus 3
\]

\[
2 \otimes^3 \cong 2 \otimes^2 \otimes 2 \cong (1 \oplus 3) \otimes 2 \\
\cong (1 \otimes 2) \oplus (3 \otimes 2)
\]
Noiseless Subsystem

Claim: There is a noiseless subsystem in three spin-$\frac{1}{2}$ particles.

Consider the irreps of rotation group.

\[
2 \otimes 2 \cong 2 \otimes 2 \cong 1 \oplus 3
\]

\[
2 \otimes 3 \cong 2 \otimes 2 \otimes 2 \cong (1 \oplus 3) \otimes 2
\]

\[
\cong (1 \otimes 2) \oplus (3 \otimes 2)
\]

\[
\cong 2 \oplus 2 \oplus 4
\]
Noiseless Subsystem

We see there are two spin-$\frac{1}{2}$ subspaces.

\[ |m = +\frac{1}{2}; \lambda = 0 \rangle = \frac{1}{\sqrt{3}} \left( |001 \rangle + \omega |010 \rangle + \omega^2 |100 \rangle \right) \]
\[ |m = -\frac{1}{2}; \lambda = 0 \rangle = \frac{1}{\sqrt{3}} \left( |110 \rangle + \omega |101 \rangle + \omega^2 |011 \rangle \right) \]
\[ |m = +\frac{1}{2}; \lambda = 1 \rangle = \frac{1}{\sqrt{3}} \left( |001 \rangle + \omega^2 |010 \rangle + \omega |100 \rangle \right) \]
\[ |m = -\frac{1}{2}; \lambda = 1 \rangle = \frac{1}{\sqrt{3}} \left( |110 \rangle + \omega^2 |101 \rangle + \omega |011 \rangle \right) \]

\[ \omega = e^{2\pi i / 3} \]
Noiseless Subsystem

We observe that

$$|m = +\frac{1}{2}; \lambda = 0\rangle = \frac{1}{\sqrt{3}} \left( |001\rangle + \omega |010\rangle + \omega^2 |100\rangle \right)$$

$$|m = -\frac{1}{2}; \lambda = 0\rangle = \frac{1}{\sqrt{3}} \left( |110\rangle + \omega |101\rangle + \omega^2 |011\rangle \right)$$

$$J_x |m = +\frac{1}{2}; \lambda = 0\rangle = |m = -\frac{1}{2}; \lambda = 0\rangle$$

$$J_z |m = +\frac{1}{2}; \lambda = 0\rangle = +\frac{1}{2} |m = +\frac{1}{2}; \lambda = 0\rangle$$

So the angular momentum operators do not change the degeneracy quantum number.
Noiseless Subsystem

To “see” the subsystem, we adopt the following perspective

$$|m; \lambda\rangle = |m\rangle \otimes |\lambda\rangle$$

$$J_x = S_x \otimes 1_\lambda$$

$$J_y = S_y \otimes 1_\lambda$$

$$J_z = S_z \otimes 1_\lambda$$

We store the qubit in the \( \lambda \) degeneracy subsystem.
Noiseless Subsystem

\[ |0\rangle_1 \]
\[ |0\rangle_2 \]
\[ |\psi_{in}\rangle_3 \]

\[ U_1 = \frac{-1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \]
\[ U_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \]
\[ U_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} \omega & -\omega \\ \omega^2 & \omega^2 \end{bmatrix} \]
Noiseless Subsystem

There is a two-dimensional NS in three spin-$\frac{1}{2}$ particles.

In Ref. [3], it was shown that there exists an $(N-1)$-dimensional NS in $N$ spin-$\frac{1}{2}$ particles.

This means in the system of four spin-$\frac{1}{2}$ particles, there is a qutrit NS.
References

