Decoherence Free Subspace & Noiseless Subsystem by example

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References

1) "Constructing qubits in physical systems", Lorenza Viola et al, J. Phys. A: Math. Gen. **34**, 7067-7079 (2001), quant-ph/0101090

2) Lorenza Viola et al, Science **293**, 2059-2063 (2001)

3) "Symmetric Construction of Reference-frame-free qudits", Jun Suzuki et al, PRA **78**, 052328, (2008), arXiv:0802.1609

Overview

1) Decoherence Free Subspace

2) Random Rotation Decoherence Model

3) Noiseless Subsystem

DFS

In QEC, we typically have

$$\rho \longrightarrow \boxed{\mathcal{E}} \longrightarrow \boxed{\mathcal{R}} \longrightarrow \mathcal{R} \circ \mathcal{E}(\rho)$$

For a given noise model, we would like to be able to undo the noise.

DFS

In QEC, we typically have

$$\rho \longrightarrow \boxed{\mathcal{E}} \longrightarrow \boxed{\mathcal{R}} \longrightarrow \mathcal{R} \circ \mathcal{E}(\rho)$$

Want to find set of states and decoding for which the following holds

$$\mathcal{R} \circ \mathcal{E}(\rho) = \rho$$

DFS

DFSs are defined as QEC codes where the decoding operation is not necessary.

$$\rho \longrightarrow \boxed{\mathcal{E}} \longrightarrow \mathcal{E}(\rho)$$

We want to find set of states satisfying

$$\mathcal{E}(\rho) = \rho$$

DFS

Example of DFS. Say we have phase noise

$$|0\rangle \xrightarrow{\mathcal{E}} |0\rangle \qquad |1\rangle \xrightarrow{\mathcal{E}} \mathrm{e}^{\mathrm{i}\phi}\,|1\rangle$$

Use this two qubit encoding

$$|0_{\mathrm{L}}\rangle = |01\rangle \qquad |1_{\mathrm{L}}\rangle = |10\rangle$$

$$\alpha\,|0_{\mathrm{L}}\rangle + \beta\,|1_{\mathrm{L}}\rangle \xrightarrow{\mathcal{E}} \mathrm{e}^{\mathrm{i}\phi}\left(\alpha\,|0_{\mathrm{L}}\rangle + \beta\,|1_{\mathrm{L}}\rangle\right)$$

Random Rotation

Say we have a spin-½ particle

$$\left|J=\tfrac{1}{2},m\right\rangle \rightarrow \mathrm{e}^{-\mathrm{i}\vec{J}\cdot\hat{n}}\left|J=\tfrac{1}{2},m\right\rangle$$

where $\hat{n}$ is a random vector.

Then can we find a DFS for this noise?

Random Rotation

We know if we have two spin-½ particles and we look at the singlet state

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Therefore $\mathrm{e}^{-\mathrm{i}\vec{J}\cdot\hat{n}} \left|s\right\rangle =$?

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Therefore $\mathrm{e}^{-\mathrm{i}\vec{J}\cdot\hat{n}} \left|s\right\rangle = \left|s\right\rangle$ for any $\hat{n}$

Random Rotation

The other triplet states $|t_-\rangle, |t_0\rangle, |t_+\rangle$ do not have the same property.

The rotation operator will mix them up. But will remain in the triplet subspace.

So the singlet spans a one-dimensional DFS. What is it good for?

How to get a two-dimensional DFS?

Random Rotation

To get a useful DFS we need four spin-½ particles.

We usually write $|s\rangle = |J=0, m=0\rangle$

$$\begin{aligned} |0_{\mathrm{L}}\rangle &= |J=0, m=0; \lambda=0\rangle \\ &= |s\rangle\,|s\rangle \\ |1_{\mathrm{L}}\rangle &= |J=0, m=0; \lambda=1\rangle \\ &= \frac{1}{\sqrt{3}}\left(|t_+\rangle\,|t_-\rangle + |t_-\rangle\,|t_+\rangle \right. \\ &\quad \left. - |t_0\rangle\,|t_0\rangle\right) \end{aligned}$$

Random Rotation

Another nicer choice of basis

$$|0_{\mathrm{L}}\rangle = \frac{1}{\sqrt{6}}\left[\omega\left(|1001\rangle + |0110\rangle\right) + \omega^2\left(|0101\rangle + |1010\rangle\right) + |0011\rangle + |1100\rangle\right] \qquad \omega = \mathrm{e}^{2\pi\mathrm{i}/3}$$

$$|1_{\mathrm{L}}\rangle = \frac{1}{\sqrt{6}}\left[\omega^2\left(|1001\rangle + |0110\rangle\right) + \omega\left(|0101\rangle + |1010\rangle\right) + |0011\rangle + |1100\rangle\right]$$

Random Rotation

Can we construct a DFS with only three spin-½ particles?

We want to find a two-dimensional subspace such that

$$\mathrm{e}^{-\mathrm{i}\vec{J}\cdot\hat{n}}\,|0_{\mathrm{L}}\rangle = |0_{\mathrm{L}}\rangle$$
$$\mathrm{e}^{-\mathrm{i}\vec{J}\cdot\hat{n}}\,|1_{\mathrm{L}}\rangle = |1_{\mathrm{L}}\rangle$$

It does not exist!

Noiseless Subsystem

Let's define subsystem first.

If we have $\mathcal{H} = \mathcal{A} \oplus \mathcal{B}$, then we say $\mathcal{A}$ is a subspace.

If we have $\mathcal{H} = (\mathcal{A} \otimes \mathcal{B})$, then we say $\mathcal{A}$ is a subsystem.

If we have $\mathcal{H} = (\mathcal{A} \otimes \mathcal{B}) \oplus \mathcal{K}$, then we say $\mathcal{A}$ is a subsystem.

All subspaces are subsystems.

Noiseless Subsystem

Decoherence free subsystem = noiseless subsystem

Recall for DFS we have $\mathcal{E}(\rho) = \rho$. For NS we have

$$\boxed{\mathcal{E}(\rho \otimes \sigma) = \rho \otimes \sigma'}$$

Noiseless Subsystem

Claim: There is a noiseless subsystem in three spin-½ particles.

Consider the irreps of rotation group.

$$2^{\otimes 2} \cong 2 \otimes 2 \cong 1 \oplus 3$$

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$$\begin{aligned}2^{\otimes 3} &\cong 2^{\otimes 2} \otimes 2 \cong (1 \oplus 3) \otimes 2 \\ &\cong (1 \otimes 2) \oplus (3 \otimes 2) \\ &\cong 2 \oplus 2 \oplus 4\end{aligned}$$

Noiseless Subsystem

We see there are two spin-½ subspaces.

$$\left|m=+\tfrac{1}{2};\lambda=0\right\rangle=\tfrac{1}{\sqrt{3}}\left(|001\rangle+\omega\,|010\rangle+\omega^2\,|100\rangle\right)$$

$$\left|m=-\tfrac{1}{2};\lambda=0\right\rangle=\tfrac{1}{\sqrt{3}}\left(|110\rangle+\omega\,|101\rangle+\omega^2\,|011\rangle\right)$$

$$\left|m=+\tfrac{1}{2};\lambda=1\right\rangle=\tfrac{1}{\sqrt{3}}\left(|001\rangle+\omega^2\,|010\rangle+\omega\,|100\rangle\right)$$

$$\left|m=-\tfrac{1}{2};\lambda=1\right\rangle=\tfrac{1}{\sqrt{3}}\left(|110\rangle+\omega^2\,|101\rangle+\omega\,|011\rangle\right)$$

$$\omega=\mathrm{e}^{2\pi\mathrm{i}/3}$$

Noiseless Subsystem

We observe that

$$\left|m=+\tfrac{1}{2};\lambda=0\right\rangle=\frac{1}{\sqrt{3}}\left(|001\rangle+\omega\,|010\rangle+\omega^2\,|100\rangle\right)$$
$$\left|m=-\tfrac{1}{2};\lambda=0\right\rangle=\frac{1}{\sqrt{3}}\left(|110\rangle+\omega\,|101\rangle+\omega^2\,|011\rangle\right)$$

$$J_x\left|m=+\tfrac{1}{2};\lambda=0\right\rangle=\left|m=-\tfrac{1}{2};\lambda=0\right\rangle$$
$$J_z\left|m=+\tfrac{1}{2};\lambda=0\right\rangle=+\tfrac{1}{2}\left|m=+\tfrac{1}{2};\lambda=0\right\rangle$$

So the angular momentum operators do not change the degeneracy quantum number.

Noiseless Subsystem

To “see” the subsystem, we adopt the following perspective

$$|m;\lambda\rangle = |m\rangle \otimes |\lambda\rangle$$

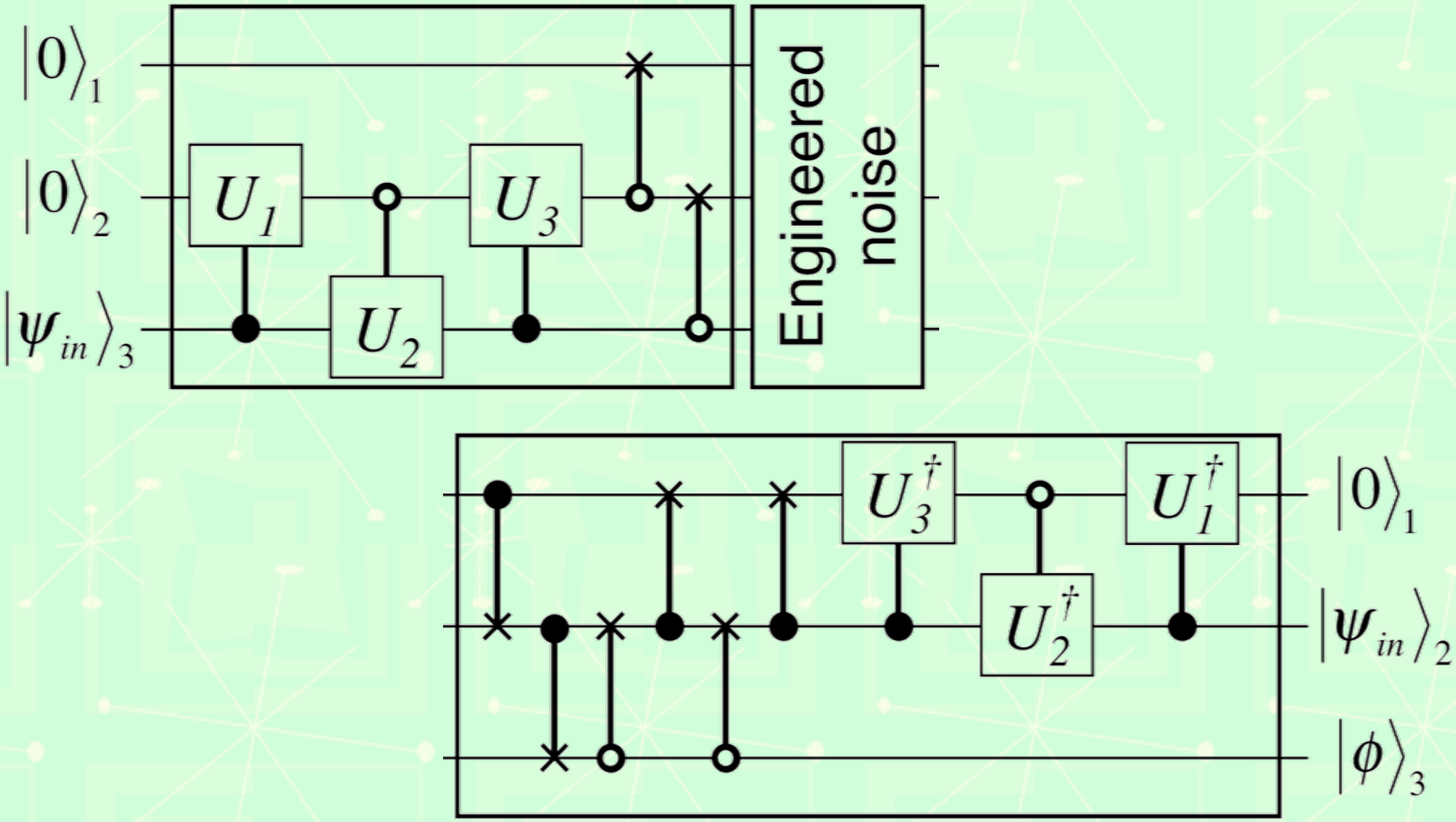
$$J_x = S_x \otimes \mathbb{1}_\lambda$$

$$J_y = S_y \otimes \mathbb{1}_\lambda$$

$$J_z = S_z \otimes \mathbb{1}_\lambda$$

We store the qubit in the λ degeneracy subsystem.

Noiseless Subsystem



$$U_1 = \frac{-1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \qquad U_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \qquad U_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} \omega & -\omega \\ \omega^2 & \omega^2 \end{bmatrix}$$

Noiseless Subsystem

There is a two-dimensional NS in three spin-½ particles.

In Ref. [3], it was shown that there exists an (N-1)-dimensional NS in N spin-½ particles.

This means in the system of four spin-½ particles, there is a qutrit NS.

References

1) "Constructing qubits in physical systems", Lorenza Viola et al, J. Phys. A: Math. Gen. **34**, 7067-7079 (2001), quant-ph/0101090

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