

Toy Model of Time-Delay Interference

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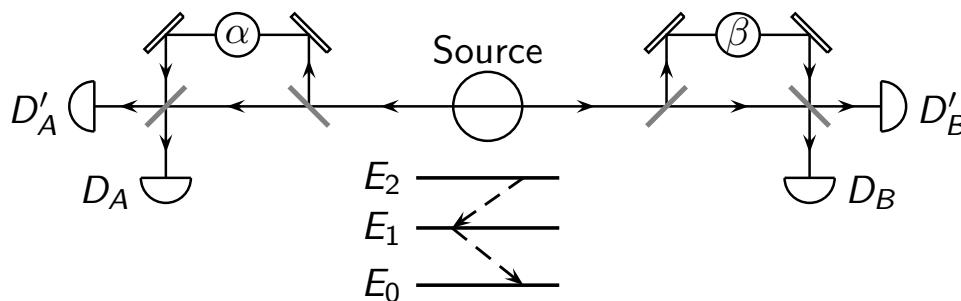
Research supported by the National Science Foundation

- References:

- J. D. Franson, "Bell inequality for position and time", *Phys. Rev. Lett.* 62 (1989) 2205-2208.

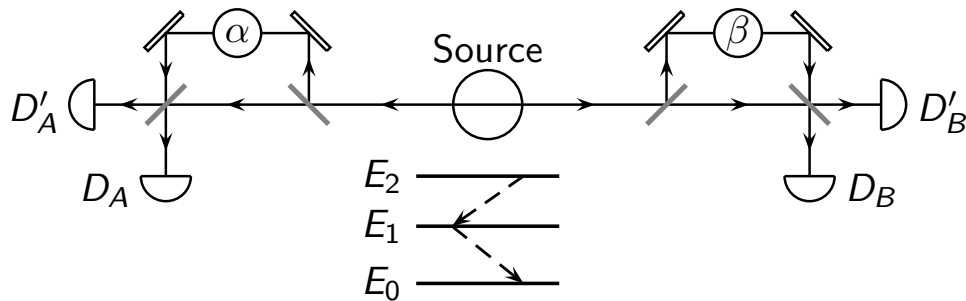
- R. B. Griffiths, *Consistent Quantum Theory* (Cambridge 2002); <http://quantum.phys.cmu.edu/CQT/>

Franson Interferometer I



- Photons emitted with energies $E_2 - E_1$ and $E_1 - E_0$
 - One goes to the left and one to the right
 - E_2 : long lifetime; E_1 : short lifetime
- Unbalanced Mach-Zehnder interferometers: Short path & long path
 - Phase shifts α and β in the longer arms
- Look for coincidences in detection:
 D_A and D_B simultaneous, or D'_A and D_B simultaneous, etc.
 - Note that D_A and D'_A will *not* be simultaneous; likewise D_B and D'_B
- Original Franson paper: source = three-level atoms. More

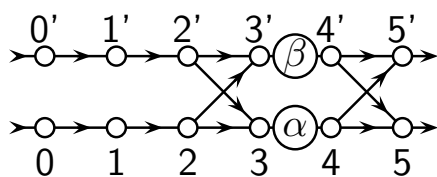
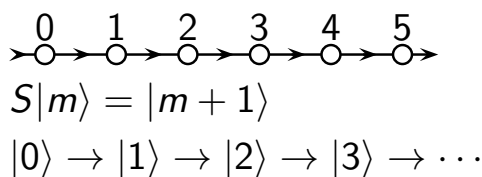
Franson Interferometer II



- Photon source in original Franson paper: three-level atoms. More recent: down conversion using pulsed pump laser.
- Original paper (1989) had title: “Bell inequality for position and time”
 - Idea: this could test the Bell inequality (1964)
 - Test had previously been carried out by Aspect (1981)

Toy Model Mach-Zehnder Interferometer I

Hopping model



$$S|m\rangle = |m+1\rangle \quad S|m'\rangle = |(m+1)'\rangle$$

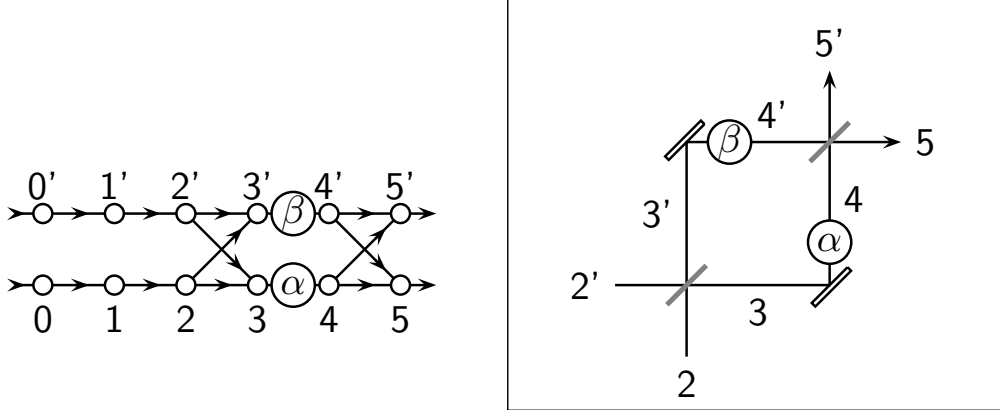
$$\sqrt{2}S|2\rangle = |3\rangle + |3'\rangle \quad \sqrt{2}S|2'\rangle = |3\rangle - |3'\rangle$$

$$S|3\rangle = \exp(i\alpha)|4\rangle \quad S|3'\rangle = \exp(i\beta)|4'\rangle$$

$$\sqrt{2}S|4\rangle = |5\rangle + |5'\rangle \quad \sqrt{2}S|4'\rangle = |5\rangle - |5'\rangle$$

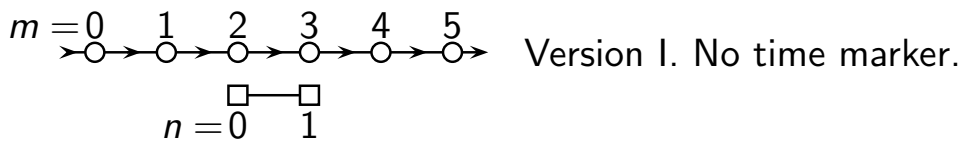
$$|2\rangle \rightarrow \frac{|3\rangle + |3'\rangle}{\sqrt{2}} \rightarrow \frac{e^{i\alpha}|4\rangle + e^{i\beta}|4'\rangle}{\sqrt{2}} \rightarrow \frac{(e^{i\alpha} + e^{i\beta})|5\rangle + (e^{i\alpha} - e^{i\beta})|5'\rangle}{2}$$

Toy Mach-Zehnder Interferometer II

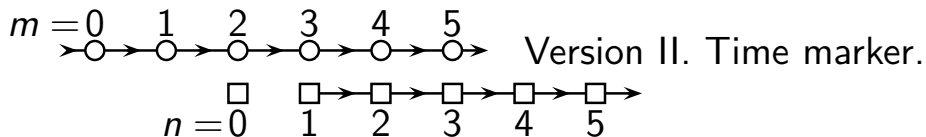


$$\begin{aligned}
 |2\rangle &\rightarrow \frac{|3\rangle + |3'\rangle}{\sqrt{2}} \rightarrow \frac{e^{i\alpha}|4\rangle + e^{i\beta}|4'\rangle}{\sqrt{2}} \rightarrow \frac{(e^{i\alpha} + e^{i\beta})|5\rangle + (e^{i\alpha} - e^{i\beta})|5'\rangle}{2} \\
 &= (e^{i\alpha}/2) [(1 + e^{i\Delta})|5\rangle + (1 - e^{i\Delta})|5'\rangle], \quad \Delta := \beta - \alpha \\
 \Pr(m = 5) &= \frac{1}{4} |1 + e^{i\Delta}|^2 = \frac{1}{2} (1 + \cos \Delta) \\
 \Pr(m = 5') &= \frac{1}{4} |1 - e^{i\Delta}|^2 = \frac{1}{2} (1 - \cos \Delta)
 \end{aligned}$$

Toy Measurement



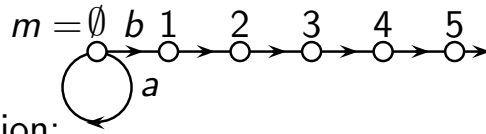
$$\begin{aligned}
 T|m, n\rangle &= RS|m, n\rangle, \quad S|m, n\rangle = |m + 1, n\rangle \\
 R|m, n\rangle &= |m, n\rangle \text{ except } R|2, n\rangle = |2, 1 - n\rangle \\
 |0, 0\rangle &\rightarrow |1, 0\rangle \rightarrow |2, 1\rangle \rightarrow |3, 1\rangle \rightarrow |4, 1\rangle \rightarrow \dots
 \end{aligned}$$



$$\begin{aligned}
 T &= RS, \quad S|m, n\rangle = |m + 1, n + 1\rangle \text{ except } S|m, 0\rangle = |m + 1, 0\rangle \\
 R|m, n\rangle &= |m, n\rangle \text{ except } R|2, 0\rangle = |2, 1\rangle \\
 |0, 0\rangle &\rightarrow |1, 0\rangle \rightarrow |2, 1\rangle \rightarrow |3, 2\rangle \rightarrow |4, 3\rangle \rightarrow |5, 4\rangle \rightarrow \dots
 \end{aligned}$$



Toy Source of Photons



- One photon production:

$|\emptyset\rangle$ = no photon present; $|m\rangle$ = photon at m .

- Time development

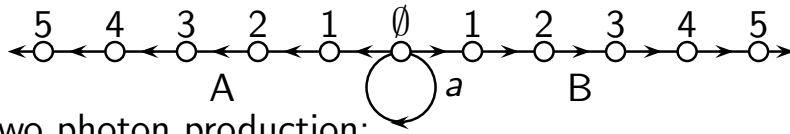
$S|m\rangle = |m+1\rangle$ except $S|\emptyset\rangle = a|\emptyset\rangle + b|1\rangle$

$|\emptyset\rangle \rightarrow a|\emptyset\rangle + b|1\rangle \rightarrow a^2|\emptyset\rangle + ab|1\rangle + b|2\rangle$

$\rightarrow a^3|\emptyset\rangle + a^2b|1\rangle + ab|2\rangle + b|3\rangle \rightarrow \dots$

- Assume that $|b| \ll 1$, $|a| \approx 1$, $a = e^{-i\epsilon}$

$|\psi_3\rangle \approx be^{-3i\epsilon}[|\emptyset\rangle + e^{i\epsilon}|1\rangle + e^{2i\epsilon}|2\rangle + e^{3i\epsilon}|3\rangle]$



- Two photon production:

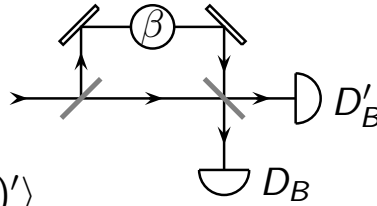
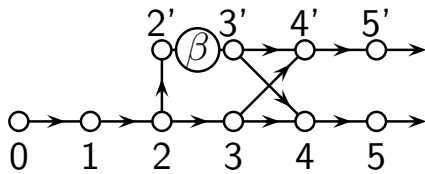
$|m\rangle_A \otimes |n\rangle_B = |m, n\rangle$; $S|m, n\rangle = |m+1, n+1\rangle$ for $m \geq 1, n \geq 1$

$S|\emptyset\rangle = a|\emptyset\rangle + b|1, 1\rangle$

$|\Psi_0\rangle = |\emptyset\rangle \rightarrow |\Psi_3\rangle \approx$

$be^{-3i\epsilon}[|0, 0\rangle + e^{i\epsilon}|1, 1\rangle + e^{2i\epsilon}|2, 2\rangle + e^{3i\epsilon}|3, 3\rangle]$

Unbalanced Mach-Zehnder



$S|m\rangle = |m+1\rangle$, $S|m'\rangle = |(m+1)'\rangle$

$S|2\rangle = \frac{|3\rangle + |2'\rangle}{\sqrt{2}}$, $S|2'\rangle = e^{i\beta}|3'\rangle$, $S|3\rangle = \frac{|4\rangle + |4'\rangle}{\sqrt{2}}$, $S|3'\rangle = \frac{|4\rangle - |4'\rangle}{\sqrt{2}}$

$|0\rangle \rightarrow |1\rangle \rightarrow |2\rangle \rightarrow (1/\sqrt{2})(|2'\rangle + |3\rangle) \rightarrow (1/\sqrt{2})e^{i\beta}|3'\rangle + (1/2)(|4\rangle + |4'\rangle)$
 $\rightarrow [e^{i\beta}(|4\rangle - |4'\rangle) + |5\rangle + |5'\rangle]/2$ No interference

$|\psi_0\rangle = (|0\rangle + e^{i\epsilon}|1\rangle)/\sqrt{2}$; $|\psi_5\rangle = S^5|\psi_0\rangle =$

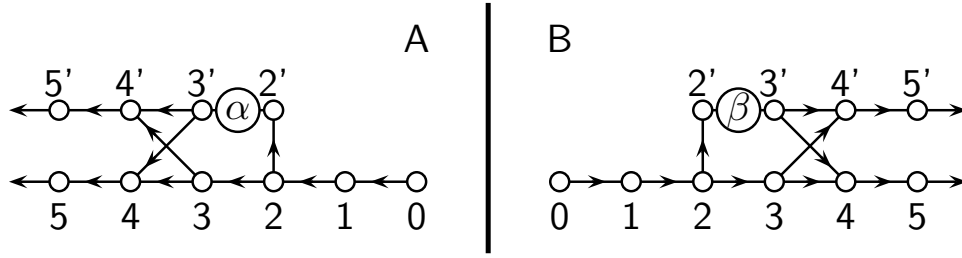
$[e^{i\beta}(|4\rangle - |4'\rangle) + (1 + e^{i(\beta+\epsilon)})|5\rangle + (1 - e^{i(\beta+\epsilon)})|5'\rangle + e^{i\epsilon}(|6\rangle + |6'\rangle)]/2\sqrt{2}$

Interference in coefficients of $|5\rangle$ and $|5'\rangle$

$\text{Pr}(5) = |1 + e^{i(\beta+\epsilon)}|^2/8 = [1 + \cos(\beta + \epsilon)]/4$

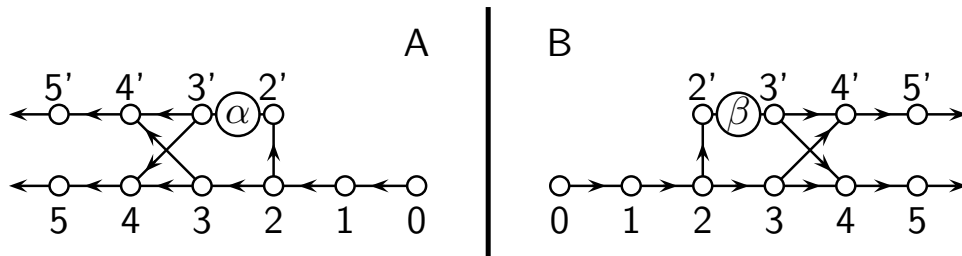
$\text{Pr}(5') = |1 - e^{i(\beta+\epsilon)}|^2/8 = [1 - \cos(\beta + \epsilon)]/4$

Toy Franson Interferometer I



- Two photons present simultaneously, so $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
States $|3\rangle_A \otimes |2\rangle_B$ write as $|3, 2\rangle$ or $|32\rangle$
Time development $T = S_A \otimes S_B$; S_A same as S_B except α vs β
 $|\Psi_0\rangle = |0, 0\rangle = |00\rangle$; $|\Psi_5\rangle = T^5|\Psi_0\rangle = [S_A^5|0\rangle \otimes S_B^5|0\rangle] = |\psi_5^\alpha\rangle \otimes |\psi_5^\beta\rangle$
 $= [e^{i\alpha}(|4\rangle - |4'\rangle) + |5\rangle + |5'\rangle] \otimes [e^{i\beta}(|4\rangle - |4'\rangle) + |5\rangle + |5'\rangle]$
 $= \{ e^{i(\alpha+\beta)}[|44\rangle - |44'\rangle - |4'4\rangle + |44\rangle] + e^{i\alpha}[|45\rangle - |4'5\rangle + |45'\rangle - |4'5'\rangle]$
 $+ e^{i\beta}[|54\rangle - |54'\rangle + |5'4\rangle - |5'4'\rangle] + [|55\rangle + |55'\rangle + |5'5\rangle + |5'5'\rangle] \} / 4$
- Is there interference? Are coincidences special?

Toy Franson Interferometer II



- $|\Phi_0\rangle = |\Psi_0\rangle + e^{i\epsilon}|\Psi_1\rangle = |00\rangle + e^{i\epsilon}|11\rangle$; $T^5|\Phi_0\rangle = |\Phi_5\rangle = |\Psi_5\rangle + e^{i\epsilon}|\Psi_6\rangle$
 $= \{ e^{i(\alpha+\beta)}[|44\rangle - |44'\rangle - |4'4\rangle + |44\rangle] + [|55\rangle + |55'\rangle + |5'5\rangle + |5'5'\rangle] +$
 $e^{i(\alpha+\beta+\epsilon)}[|55\rangle - |55'\rangle - |5'5\rangle + |5'5'\rangle] + e^{i\epsilon}[|66\rangle + |66'\rangle + |6'6\rangle + |6'6'\rangle]$
 $+ \text{non-coincidence terms} \}$

- Interference terms in the 55, 55' coincidences:

$$[1 + e^{i(\alpha+\beta+\epsilon)}](|55\rangle + |5'5'\rangle) + [1 - e^{i(\alpha+\beta+\epsilon)}](|55'\rangle + |5'5\rangle)$$

$$\text{Pr}(55) = \text{Pr}(5'5') \propto 1 + \cos(\alpha + \beta + \epsilon); \quad \text{Pr}(55') = \text{Pr}(5'5) \propto 1 - \cos(\alpha + \beta + \epsilon)$$