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Histories and Consistency

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References:

CQT = *Consistent Quantum Theory* by Griffiths (Cambridge, 2002), Chs. 8, 9, 10, 11.

1 Histories

1.1 Classical stochastic processes

★ When probability theory is applied to a sequence of events in time, the sample space consists of what we shall call *histories*. For example, if a coin is tossed twice in a row, the four histories HH, HT, TH, and TT, where HT stands for "heads on the first toss and tails on the second," represent mutually exclusive possibilities, one and only one of which will occur in a single experimental run. The event algebra contains 16 "events," including "heads on the first toss," "the same both times," and the like.

• The sample space of histories for N tosses of one coin is identical to the sample space of outcomes when N coins are tossed at the same time. The same analogy works for quantum systems.

1.2 Quantum histories

★ Quantum histories consist of a sequence of events at a series of times, but they are now specified by giving a projector on the quantum Hilbert space at each time. Thus for a set of times

$$t_1 < t_2 < \cdots t_f \tag{1}$$

one can specify a set of projectors $F_1, F_2, \ldots F_f$, whose significance is that the system of interest has (or had) the property F_j at the time t_j.

• For example, in the case of a qubit we might have $F_1 = [0]$, $F_2 = [1]$, $F_3 = [1]$, where $[\psi] = |\psi\rangle\langle\psi|$ for a normalized state $|\psi\rangle$.

★ A useful analogy. Consider a composite system consisting of a number of individual systems, where the total Hilbert space is the tensor product of Hilbert spaces for the individual systems:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \mathcal{H}_f \tag{2}$$

• The projector for a product property of such a composite system can then be written as

$$F = F_1 \otimes F_2 \otimes \cdots F_f. \tag{3}$$

The significance of F is that system 1 has property F_1, system 2 property F_2, etc.

★ Following the analogy we define the *histories Hilbert space*

$$\breve{\mathcal{H}} = \mathcal{H}_1 \odot \mathcal{H}_2 \odot \cdots \mathcal{H}_f. \tag{4}$$

◦ Here $\odot$ indicates a tensor product, the same as $\otimes$; using a distinct symbol in the case of histories is convenient but not essential.

◦ The subscript labels on the right can be dropped (or retained if one finds them helpful), as we shall assume that $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_f = \mathcal{H}$, where $\mathcal{H}$ is just the "ordinary" Hilbert space of the system of interest, used when describing its properties at a single time. Thus if the system is a qubit and $f = 5$, $\mathcal{H}$ is a 2-dimensional Hilbert space and $\breve{\mathcal{H}}$ a 32-dimensional Hilbert space.

★ The projector on the Hilbert space $\breve{\mathcal{H}}$ for the history in which F_1 occurs at t_1, F_2 occurs at t_2, etc., is

$$Y = F_1 \odot F_2 \odot \cdots F_f, \tag{5}$$

• The use of the $\odot$ symbol is a convenience, not a necessity. If one has a bipartite system $\mathcal{H}_a \otimes \mathcal{H}_b$ one could use either

$$A_1 \otimes B_1 \odot A_2 \otimes B_2 \odot A_3 \otimes B_3 ... \quad \text{or} \quad A_1 \otimes B_1 \otimes A_2 \otimes B_2 \otimes A_3 \otimes B_3, \tag{6}$$

for a history involving three product events at successive times, but the left side makes things a bit more obvious.

1.3 Histories sample space

★ A quantum sample space is always a decomposition of the identity: a collection of projectors that sums to the identity. For histories the Hilbert space is $\breve{\mathcal{H}}$, (4), and we denote the identity as $\breve{I}$, to clearly distinguish it from the identity I for the Hilbert space $\mathcal{H}$ of the system at a single time.

• In what follows we shall only employ projectors of the product form, as in (5). It may be that more general projectors are useful, but product projectors seem sufficient for most physical applications.

★ A particularly simple (and useful) approach to generating history projectors is to assume that at time t_j there is a decomposition of the identity

$$I_j = \sum_{\alpha_j} P_j^{\alpha_j}. \tag{7}$$

Here the subscript j labels the time, and I_j can be thought of as the identity operator for the Hilbert space $\mathcal{H}_j$ on the right side of (4). Of course I_j is the same for every j, as it is just the identity operator on $\mathcal{H}$, but subscripting it helps to emphasize that we allow different decompositions of the identity at different times.

• As time is denoted by a subscript, it is useful to employ the superscript position on the right side of (7) to label the projector. Since any projector is equal to its square we really do not need to reserve the superscript position for an exponent.

• The projectors

$$Y^\alpha = P_1^{\alpha_1} \odot P_2^{\alpha_2} \cdots \odot P_f^{\alpha_f}, \tag{8}$$

where α is an f-tuple of labels

$$\alpha = (\alpha_1, \alpha_2, \ldots \alpha_f), \tag{9}$$

comprise a series of projectors that form a decomposition of $\breve{I}$ and make up a histories sample space.

★ Examples

• Example 1. A qubit at two times t_1 and t_2.

$$I_1 = [0] + [1], \quad I_2 = [0] + [1]. \tag{10}$$

Four separate histories: Y^{00}, Y^{01}, Y^{10}, Y^{11}, in the notation of (8).

∘ Note that the product

$$Y^{00} Y^{01} = ([0] \odot [0]) \cdot ([0] \odot [1]) = 0 \tag{11}$$

by the usual rule that $(A' \otimes B') \cdot (A \otimes B) = A'A \otimes B'B$. In a similar way $Y^\alpha Y^\beta = 0$ whenever $\alpha \neq \beta$.

∘ Note that $Y^{00} + Y^{01} = [0] \odot I$. Proceeding in this fashion one can check that the sum of the four Y^α is $\breve{I} = I \odot I$.

• Example 2. A qubit at two times t_1 and t_2.

$$I_1 = [0] + [1], \quad I_2 = [+] + [-]. \tag{12}$$

∘ This includes histories such as $[0] \odot [-]$, whose physical interpretation for a spin half particle is $S_z = +1/2$ at t_1 AND $S_x = -1/2$ at t_2.

∘ Note that the assertion $S_z = +1/2$ AND $S_x = -1/2$ at a *single* time is quantum nonsense, but as long as the properties are assigned at *different* times there is no problem. It is similar to "$S_z = +1/2$ for particle a and $S_x = -1/2$ for particle b."

□ Exercise. Check that the Y^α obtained from (12) are mutually orthogonal, and that they add up to $\breve{I}$.

★ Sometimes it is helpful to use the following notation for a sample space of histories of the type under consideration:

$$\{P_1^1, P_1^2 \ldots\} \odot \{P_2^1, P_2^2 \ldots\} \odot \cdots. \tag{13}$$

Thus the sample space for Example 2 can be written compactly as

$$\{[0], [1]\} \odot \{[+], [-]\} \tag{14}$$

in a notation that tells us which decompositions of the identity are involved at each time, and lets us read off the four histories in a convenient way.

★ The rules for compatibility of different sample spaces (families) of histories are the same as for other decompositions of the identity. If the projectors from a family $\{Y^\alpha\}$ commute with those from a family $\{Z^\beta\}$, $Y^\alpha Z^\beta = Z^\beta Y^\alpha$ for every α and β, the families are compatible; otherwise they are incompatible and cannot be combined.

• If the two families $\{Y^\alpha\}$ and $\{Z^\beta\}$ involve different sets of times, $t_1 < t_2 < \cdots t_f$ for $\{Y^\alpha\}$ and $t'_1 < t'_2 < \cdots t'_g$ for $\{Z^\beta\}$, one uses the following trick. Take the union of the two sets of times to form a single set $t''_1 < t''_2 < \cdots t''_h$, and for each family introduce at each time missing from this larger list the single property consisting of the identity operator I.

• One can always introduce I at additional times without altering the physical contents of a history, because the projector I tells one nothing at all: it is the property that always happens or is always true, and therefore implies nothing.

• Example. For the $\{Y^\alpha\}$ family suppose that $t_1 = 1.5$, $t_2 = 2.0$, and the histories are based upon the same decomposition of the identity $I = P + \tilde{P}$ at the two times. For $\{Z^\beta\}$ use $t'_1 = 0.0$, $t'_2 = 2.0$, and employ $I_1 = Q + \tilde{Q}$ at t'_1 and $I_2 = R + \tilde{R}$ at t'_2. The combined set of times is $t''_1 = 0.0$, $t''_2 = 1.5$, $t''_3 = 2.0$. The history $P \odot \tilde{P}$ in the $\{Y^\alpha\}$ family is the same as $I \odot P \odot \tilde{P}$ for the new set of three times, whereas $Q \odot \tilde{R}$ in the $\{Z^\beta\}$ family is now $Q \odot I \odot \tilde{R}$.

□ Exercise. Show that the $\{Y^\alpha\}$ and $\{Z^\beta\}$ families in this example are compatible if and only if $RP = PR$. In the case in which they are compatible, which histories form a basis for the common refinement?

★ A type of sample space which turns out to be very useful in practice is based upon a *fixed initial state* $|\psi_0\rangle$ at time t_0. (One could just as well call the initial time t_1.) The history projectors are

$$Y^\alpha = [\psi_0] \odot P_1^{\alpha_1} \odot P_2^{\alpha_2} \cdots \odot P_f^{\alpha_f}; \quad Y^0 = (I - [\psi_0]) \odot I \odot I \cdots \odot I. \tag{15}$$

• Here the special history Y^0 is present in order to complete the sample space: the decomposition of the history identity is

$$\breve{I} = Y^0 + \sum_\alpha Y^\alpha. \tag{16}$$

However, since we are interested only in histories which start with the initial state $|\psi_0\rangle$, or $[\psi_0]$, we assign zero probability to Y^0—it never occurs—which means that the probabilities associated with the histories Y^α with initial state $[\psi_0]$ sum up to 1.

∘ The Born rule with a specific starting state $|\psi_0\rangle$ employs a sample space of this type.

★ Histories based upon a set of mutually exclusive events at every time, (7), or at every time following the initial $[\psi_0]$ at t_0, (15), are not the only possibilities; other kinds arise in various applications. See CQT Sec. 14.4 for an example.

2 Probabilities

2.1 Introduction

★ Only in particular circumstance do the laws of quantum mechanics, as currently understood, allow one to assign probabilities to families of histories.

• Schrödinger's equation can only be used to generate the unitary time development operators $T(t', t)$ when the system is *closed* or *isolated*.

• In cases in which a system is not isolated from the outside world (or "environment") it is helpful to start by applying Schrödinger's equation to the combined system plus environment regarded as forming a big, isolated system. The time development operators can then be used, at least in certain cases, to assign probabilities to the subsystem one is interested in.

◦ In practice it may be impractical to integrate Schrödinger's equation for a large, complicated system, but understanding what should be done in principle helps in making reasonable approximations.

2.2 Consistency using chain kets

• If only two times are involved, the Born rule and its various generalizations can be used to assign probabilities to a closed system. See separate notes on the Born rule.

★ For three or more times one has to use extensions of the Born rule, and these extensions require that certain *consistency conditions* (also known as *decoherence conditions*) be satisfied before probabilities can be assigned.

• For an extensive discussion of consistency conditions see Chs. 10 and 11 of CQT. These notes are too brief to give a full discussion of the rules, but we'll discuss a particularly useful case.

★ The easiest situation in which to apply consistency conditions and compute probabilities for multiple-time sample spaces is when the histories all involve a fixed initial state $|\psi_0\rangle$ at time t_0 and the sample space is of the form (15).

• We define *chain kets*

$$|\alpha\rangle = |(\alpha_1, \alpha_2, \dots \alpha_f)\rangle = P_f^{\alpha_f} T(t_f, t_{f-1}) P_{f-1}^{\alpha_{f-1}} T(t_{f-1}, t_{f-2}) \cdots P_1^{\alpha_1} T(t_1, t_0) |\psi_0\rangle, \tag{17}$$

one for each history in the sample space.

◦ Warning. Note that the α_j appear (unfortunately) in opposite orders on the two sides of this equation.

★ The *consistency conditions* are then the requirement that

$$\langle\alpha|\beta\rangle = 0 \text{ for } \alpha \neq \beta, \tag{18}$$

where $\alpha = \beta$ means $\alpha_j = \beta_j$ for $j = 1, 2, \dots f$, and otherwise $\alpha \neq \beta$. Note that (18) is just the requirement that the set $\{|\alpha\rangle\}$ of chain kets be an *orthogonal* collection.

★ History sample spaces can only be combined, Sec. 1.3, when the corresponding decompositions of the history identity $\breve{I}$ are compatible. However, even when this is the case and when both of the families satisfy the consistency conditions it may still happen that the common refinement, whereas it constitutes a proper decomposition of the identity, does not satisfy itself satisfy the consistency conditions. In this case there is no way to assign probabilities to the refinement, and in this sense the original families are incompatible with each other.

• The *single framework rule* that forbids combining incompatible decompositions of the identity also rules out combinations of the type under discussion.

∘ A simple example. Consider 1 qubit with trivial dynamics, $T(t',t) = I$, and the two families using the same initial state [0]:

$$\{Y^\alpha\} = [0] \odot \{[0],[1]\} \odot I, \quad \{Z^\beta\} = [0] \odot I \odot \{[+],[-]\} \tag{19}$$

□ Exercise. Convince yourself that both families satisfy the consistency conditions. Then find the common refinement and show that it is not consistent.

2.3 Probabilities using chain kets

★ When the consistency conditions (18) are satisfied probabilities are assigned to the histories using the following generalization of the Born rule:

$$\Pr(\alpha) = \langle\alpha|\alpha\rangle, \tag{20}$$

where we could also write $\Pr(\alpha|\psi_0)$ to emphasize that these probabilities are conditional on the initial state $|\psi_0\rangle$

• Probabilities for compound histories corresponding to the sum of two or more of the Y^α are, as usual given by adding the probabilities of the individual histories as computed using (20).

★ Example. For a single qubit assume that $|\psi_0\rangle = |1\rangle$, $T(t_1,t_0) = H$, $T(t_2,t_1) = X$. Further assume the decompositions $I_1 = I_2 = [0] + [1]$ at times t_1 and t_2. The chain kets as defined in (17) are then

$$|(0,0)\rangle = 0, \quad |(1,1)\rangle = 0, \quad |(0,1)\rangle = |1\rangle/\sqrt{2}, \quad |(1,0)\rangle = -|0\rangle/\sqrt{2}, \tag{21}$$

where, e.g., $|(0,1)\rangle$ corresponds to the history $[1] \odot [0] \odot [1]$.

□ Exercise. Check the correctness of (21).

• It is then obvious by inspection that the four chain kets, two of which are 0, are mutually orthogonal. So the consistency conditions are satisfied. There are only two histories with nonzero probability, namely

$$[0] \odot [0] \odot [1], \quad [0] \odot [1] \odot [0], \tag{22}$$

each of which has probability 1/2.

★ As an example of an *inconsistent* family for which the consistency conditions are *not* satisfied, use the preceding example with $T(t_2,t_1) = H$ instead of X. To see that consistency fails, it suffices to note that

$$|(0,0)\rangle = \frac{1}{2}|0\rangle, \quad |(1,0)\rangle = -\frac{1}{2}|0\rangle, \tag{23}$$

and these kets are clearly not orthogonal to each other. Hence we are unable to assign probabilities using our extension of the Born rule.

□ Exercise. Check (23) and derive the other two chain kets.