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Measurements

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References:
CQT = *Consistent Quantum Theory* by Griffiths (Cambridge, 2002)
QCQI = *Quantum Computation and Quantum Information* by Nielsen and Chuang (Cambridge, 2000).

Contents

1 Introduction

1.1 Scope of these notes

★ In quantum foundations, the study of the conceptual basis of quantum mechanics, "measurements" are an enormous conceptual headache. They have given rise to endless arguments.

• To resolve the arguments one needs to adopt a consistent approach in which the measuring apparatus itself is treated in quantum mechanical, not classical, terms, and probabilities are properly defined. This is discussed at some length in CQT Chs. 17 and 18. However, the reader who wants to delve into this would do well to begin with toy models in Sec. 7.4 of CQT.

• The axiomatic approach found in QCQI can be regarded as a recipe for making calculations. When properly applied it gives the right answers. The danger comes from mistaking calculational rules, such as wave function collapse, for real physical processes. This mistake leads to all sorts of magic, mystery, and misguided notions of what the quantum world is like (e.g., the idea of long-range superluminal influences).

• The purpose of these notes is not to discuss measurements in detail. For that see CQT Chs. 17, 18. Instead, it is to explain how measurements are to be understood in the context of quantum information.

1.2 Measurements and histories

★ A measuring apparatus determines properties of a quantum system by interacting with it in some way. Consequently, it is impossible to treat either the system or the apparatus as isolated, at least during the crucial time period when they interact. Instead, one should regard both together as constituting a single combined and isolated quantum system to which Schrödinger's equation or the corresponding unitary time development operators apply.

★ It is then necessary to introduce appropriate families of quantum histories in order to describe what is going on. We need families with the following features:

• At a time when the measurement are completed a suitable decomposition of the identity for the apparatus in which *measurement outcomes* are well defined.

◦ In the antique terminology employed in quantum foundations, where these issues were discussed well before the electronic age, measurement outcomes are traditionally referred to as *pointer positions*. However, any sort of macroscopically distinct states will do equally well. Symbols printed out on a sheet of paper. Bits stored in the computer memory, etc.

◦ The failure on the part of physicists to introduce families with projectors of this type is one of the main reason for a lack of progress on the "measurement problem" in quantum foundations.

• Next, in order to speak of the measurement as *having measured something* the family of histories must contain projectors corresponding to appropriate microscopic physical properties of the measured system before the measurement took place.

◦ Textbook quantum mechanics does not contain the concepts needed to deal with properties of the measured system before a measurement takes place, and this is the reason why the computational rules (including those found in QCQI) appear so mysterious.

• Finally the families must contain some sort of initial state(s) connected with the way in which the apparatus was set up, the way in which the quantum system of interest was prepared, etc.

★ Consequently, any adequate discussion of measurements using basic quantum principles rather than arm waving must be capable of addressing histories containing at least three times: the initial time of preparation, the time just before the measurement takes place when the system possesses the measured property, and the time after the measurement has taken place when it is important to know how the macroscopic "pointer position" is correlated with the earlier microscopic property one is interested in.

2 Destructive Measurements

2.1 Standard measuring device

★ In quantum circuits the simplest measurement device can be indicated by a symbol shaped like the letter D; see Fig. 1.

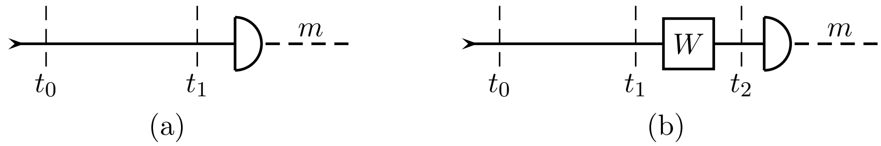

Figure 1: (a) Detector for measuring qubit in standard basis. (b) Detector for measuring qubit in the $\{|w^+\rangle, |w^-\rangle\}$ basis.

◦ In QCQI the device is indicated by a dial with a pointer placed in a rectangular box. This looks better than a D and embodies the grand tradition of pointer positions. Alas, authors in a hurry to publish papers, students rushing to complete their homework, and professors anxious to make up the next assignment don't have time for this fancy stuff. The D is perfectly adequate and by now fairly standard.

★ The D terminates the horizontal line representing a qubit (or possibly a qudit), indicating that this object is no longer available at later times; it has disappeared. For this reason we refer to this as a *destructive* measurement.

• In practice typical laboratory measurements of the sort that are interesting for quantum information are usually destructive. A photon is eaten up by the photodetector. The state of the ion in the trap after being irradiated by a strong laser pulse need bear no simple relationship to the earlier property that was being measured. So while the ion is still there, the information it used to carry is no longer there.

★ The *measurement outcome* is indicated in Fig. 1 by a dashed line, and is referred to as a "classical bit" if the measurement is carried out on a qubit.

◦ It would be far better to indicate the measurement outcome by a heavy solid line, or a double line as in QCQI, because the outcome is stored in a macroscopic state which, while still quantum mechanical, is much more robust that the original qubit, and can be thought of as existing in multiple copies. Again, the dashed line is simply a consequence of a decaying culture in which the professors and the students are always in too much of a hurry. . . .

• The dashed lines in Fig. 1(a) carry a label m, where $m = 0$ or $m = 1$ are the two measurement outcomes. We shall refer to these outcomes as $D = 0$ and $D = 1$.

2.2 Competent experimentalist principle (CEP)

• As discussed above in Sec. 1.2, a complete complete quantum discussion of how the outcome in Fig. 1(a) is related to quantum properties of the qubit at time t_1 just before the measurement took place requires that the apparatus, always itself macroscopic and hence from a quantum perspective very complicated, be included in the analysis. See CQT Chs. 17 and 18 for the basic principles. They can be used to justify the discussion which follows.

★ *Competent Experimentalist Principle* or CEP. Apparatus built by a competent experimentalist and intended to measure some microscopic property will do what it was designed to do.

• This means in particular that if we analyze the apparatus and system to be measured in quantum mechanical terms using an appropriate pointer decomposition of the identity, referred to somewhat inaccurately as the "pointer basis," for the macroscopic apparatus at a time when the measurement is over, and the appropriate "property basis" for the system at a time just before the measurement takes place there will be a perfect statistical correlation between the two.

• In particular, if the system entering the apparatus is in one of the basis states $|a^k\rangle$, $k = 0, 1, \cdots$, then a short time later the apparatus pointer will be in position k corresponding to the projector

P^k, which is to say

$$\Pr(P^j \,|\, a^k) = \delta_{jk}. \tag{1}$$

★ Suppose that our knowledge of the property of the measured system some time before the measurement took place is embodied in an initial $|\psi_0\rangle$ at time t_0, and by a proper quantum mechanical calculation (the Born rule or some extension) we determine that the probability the system will be in the state/have the property $|a^k\rangle$ at the time t_1 just before it interacts with the apparatus is

$$p_k = \Pr(a^k \,|\, \psi_0). \tag{2}$$

Then we can combine this with (1) to obtain

$$\Pr(P^j \,|\, \psi_0) = \sum_k \Pr(P^j \,|\, a^k) \Pr(a^k \,|\, \psi_0) = p_j. \tag{3}$$

○ The first equality in (3) is obtained by writing

$$\Pr(P^j \,|\, \psi_0) = \sum_k \Pr(P^j \,|\, a^k, \psi_0) \Pr(a^k \,|\, \psi_0), \tag{4}$$

which is a formula of standard probability theory, and then making the additional assumption that

$$\Pr(P^j \,|\, a^k, \psi_0) = \Pr(P^j \,|\, a^k). \tag{5}$$

○ The intuitive meaning of (5) is that the probabilities referring to the pointer positions are determined once a $|a^k\rangle$ is known, and not influence by $|\psi_0\rangle$. The competent experimentalist has constructed his apparatus in such a way that there are no additional influences beyond the state of the particle when it enters the detector. Designing things so as to eliminate such extraneous influences, and then carrying out reasonable checks to show that they are absent, is part of his job.

☐ Exercise. Check that (4) is a sound principle of probabilistic reasoning, and then show that it plus (5) leads to the first equality in (3).

• The conclusion upon comparing (2) and (3) is that the *probability of a measurement outcome* (for apparatus built by a competent experimentalist) is identical to the *probability that the measured system had the corresponding property* just prior to when the measurement took place.

★ So why all the talk about measurements when all we need to calculate is the *microscopic* probability (2)? It has to do with the historical development of quantum mechanics: "measurement" was invoked to get rid of certain conceptual problems and paradoxes back in the days before there was a fully consistent probabilistic formulation of quantum theory.

• In any case there is nothing wrong with mentioning measurements. The student asked on an examination to "calculate the probability that a measurement of this particular sort has the following outcome" should assume (in the absence of any indication to the contrary) an implicit reference to the CEP, and then go ahead and use the Born rule (or whatever) to find the probability of the property (properties) of interest just before the measurement took place.

2.3 Examples

★ Consider the circuit in Fig. 1(a), and assume that the initial state at t_0 is

$$|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{6}$$

with $|\alpha|^2 + |\beta|^2 = 1$, so $|\psi_0\rangle$ is normalized. What is the probability that the measurement will yield $D = 0$ or 1?

• The device has been designed to determine the properties in the decomposition $[0] + [1] = I$, so all we need to do is to calculate their probabilities using the Born rule. Since the dynamics is trivial, $|\psi_1\rangle = |\psi_0\rangle$:

$$\Pr(m = 0) = \langle\psi_1|[0]|\psi_1\rangle = |\alpha|^2, \quad \Pr(m = 1) = \langle\psi_1|[1]|\psi_1\rangle = |\beta|^2. \tag{7}$$

★ Of course a competent experimentalist who can build a device to measure whether a qubit is in $|0\rangle$ or $|1\rangle$ can also build one to measure whether a system is in $|w^+\rangle$ or $|w^-\rangle$ for any pair of states that form an orthonormal basis for one qubit. However, there is an alternative to building all manner of different devices, and it is represented in Fig. 1(b). Namely, by inserting a suitable one-qubit gate, the unitary

$$W = |0\rangle\langle w^+| + |1\rangle\langle w^-| \tag{8}$$

in front of the standard measurement device one can arrange for a measurement in whatever basis one wants.

☐ Exercise. Show that W in (8) is unitary, and that *any* one qubit unitary can always be written in the form (8) using a suitable pair of orthonormal states $\{|w^+\rangle, |w^-\rangle\}$. [Hint. How can one characterize a unitary operator by its action on the kets that comprise an orthonormal basis? One may prefer to think about $W^\dagger$ rather than W.]

• Now consider the family of histories at times $t_0 < t_1 < t_2$:

$$[\psi_0] \odot \{[w^+], [w^-]\} \odot \{[0], [1]\}. \tag{9}$$

It is easy to show that there are only two nonzero chain kets, and they are orthogonal to each other. Further, the corresponding probabilities are

$$\Pr([\psi_0] \odot [w^+] \odot [0]) = |\langle w^+|\psi_0\rangle|^2, \quad \Pr([\psi_0] \odot [w^-] \odot [1]) = |\langle w^-|\psi_0\rangle|^2. \tag{10}$$

☐ Exercise. Check the math.

• Consequently, it follows that the measurement outcome is $D = 0$ if and only if the qubit had the property $[w^+]$ at time t_1, and $D = 1$ if and only if it had the property $[w^-]$ at t_1. So we can think of the unitary W together with D in Fig. 1(b) as comprising a slightly more complicated measuring device which measures whether the qubit is in the state $|w^+\rangle$ or $|w^-\rangle$ at the time t_1 just before it enters the device.

★ The preceding strategy has an obvious generalization to the case of two (or more) qubits, as shown in Fig. 2(a), where prefixing a 2 qubit unitary operator W to a pair of standard detectors D_a and D_b makes it possible to carry out a measurement in any basis of the 4 dimensional two-qubit tensor product space by simply choosing W in an appropriate way. If one is interested in a basis which is a product of bases for $\mathcal{H}_a$ and $\mathcal{H}_b$, it suffices to have W a product of two unitaries, as in Fig. 2(b).

2.4 Using the right basis

★ CharliE is a Competent Experimentalist who knows how to build apparatus in such a way that it can reliably measure some microscopic property. He has just constructed a device that will measure S_x for a spin-half particle; $S_x = +1/2$ and $-1/2$ correspond to the the qubit states $|+\rangle$ and $|-\rangle$. Naturally, he also knows how to build an apparatus that will measure S_z, for which $+1/2$ and $-1/2$ correspond to $|0\rangle$ and $|1\rangle$.

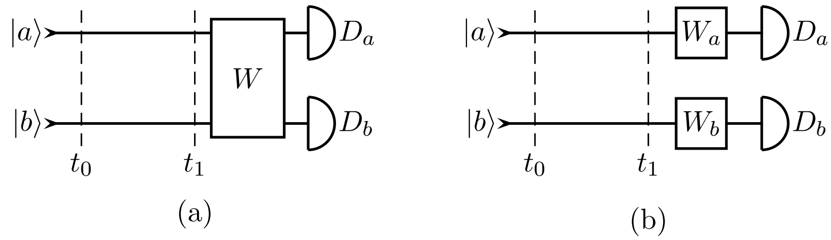

Figure 2: Two qubit detector for: (a) General measurement basis; (b) Product of bases.

★ Charlie collaborates with a *Helpful Theorist* HerberT, who uses the Born rule (or sometimes its extensions, see CQT Ch. 10) to calculate the probability that if the qubit is in the state $|\psi_0\rangle$ at the beginning of the experiment, and then undergoes various interactions on the way to the final measurement, it will be in one of the two states $|0\rangle$ or $|1\rangle$ just before it reaches Charlie's detector.

• The two get together after a long night of hard work.

Charlie: What a tiresome business! I worked all night and only managed to make a thousand measurements of S_x. Nonetheless, that's enough so I can be confident that $\Pr(S_x = +1/2)$ is 0.33, and $\Pr(S_x = -1/2)$ is 0.67, given that I always used the starting state $|\psi_0\rangle$.

Herbert: What a nasty calculation. It took me all night because that system of yours is so complicated, but finally I figured out that the proper quantum calculation yields $\Pr(S_z = +1/2)$ is 0.82, and $\Pr(S_z = -1/2)$ is 0.18. Doesn't look like that agrees with what you measured.

Charlie: But you calculated the probabilities for S_z, not S_x. That wasn't very helpful!

Herbert: Oh come on. Why didn't you measure S_z rather than S_x?

• It is clear that they've had a long night, and Herbert is not being too helpful, so let's send them out for a few cups of coffee. After the break the discussion is more civil:

Charlie: Tell you what. During the next run I'll sometimes measure S_x and sometimes measure S_z, and I'll keep the data separate. And why don't you go ahead and calculate the S_x probabilities as well, so we can make a comparison in both cases.

Herbert: Agreed! See you tomorrow.

★ The moral to draw from this is that the theorist, if he wants to be helpful to the competent experimentalist who has built an apparatus to measure something, needs to adopt for his calculations a quantum sample space that includes, at the time just before the measurement takes place, the properties that the apparatus is designed to measure. There is no rule that says the theorist must do this, and of course it does no harm to calculate things in all sorts of incompatible frameworks. But the properties most directly correlated with measurement outcomes are the ones the competent experimentalist designed his apparatus to measure, the $\{|a^k\rangle\}$ employed in (1).

★ Consequently, when simple measurements are under discussion in the context of quantum information, we will make the assumption that the basis (more generally, decomposition of the identity) adopted for the microscopic system corresponds to those properties the competent experimentalist wanted to measure.

2.5 The counterfactual error

• The following type of reasoning has a long history in quantum foundations, and has led to all sorts of confusion.

★ Suppose Charlie's apparatus is set up to measure S_z and, to be specific, yields the outcome (pointer position) corresponding to $S_z = +1/2$. But suppose that in this particular case, involving this particular particle, Charlie had instead used the S_x rather than the S_z apparatus.

◦ Sometimes the apparatus can be designed so that it has a switch with two positions, with the switch in the "z" position it measures S_z, and with the switch in the "x" position it measures S_x. Changing the unitary W in Fig. 1(b) can do the switching. Charlie might choose one position or the other just before the arrival of the particle.

• Then surely the S_x apparatus would have yielded a pointer position corresponding to one of the two values, $S_x = +1/2$ or $S_x = -1/2$, just before the measurement took place.

• Consequently, in this particular case when we know (using the Competent Experimentalist Principle) that S_z was $+1/2$ just before the measurement, we conclude, via a suitable exercise of imagination, that if Charlie had instead measured S_x it too would have had a definite value, $+1/2$ or $-1/2$. Thus we conclude that a particle must always have not only a definite S_z value (the one the measurement is going to reveal), but also a definite S_x value which the counterfactual measurement would have revealed.

◦ The general idea of a *counterfactual* is to imagine a world similar to the actual world but different in some respects, and then try and argue about what *would* have happened in this imaginary world.

★ To be sure, the Hilbert space is not big enough to hold both an S_z and an S_x value for a spin-half particle. So what should we do? There are various approaches:

• One can try and enlarge the Hilbert space by adding "hidden variables." Approaches of this sort have been tried. The best known is due to de Broglie and Bohm, and leads to various odd results, such as an instantaneous action-at-a-distance that is difficult to reconcile with relativity.

• One can publish lengthy papers and even books about quantum paradoxes, thus keeping up interest in the field while never solving the outstanding problems.

• Or one can look for problems in the logical structure of counterfactual reasoning as applied to quantum systems. When this is done, see Ch. 19 of CQT, one finds that in cases where it gives rise to paradoxes, counterfactual reasoning always involves a violation of the single framework rule of quantum reasoning: incompatible sample spaces are being combined, or consistency conditions ignored.

◦ In the case at hand it is pretty obvious that the reasoning involves combining incompatible sample spaces, since this is the only way to reach the nonsensical conclusion that a given spin-half particle has both an S_x and an S_z value at the same time.

• Conclusion: Beware of counterfactuals in quantum reasoning. There are correct as well as incorrect ways of using them. See Ch. 19 of CQT for more details.

3 Partial measurements

3.1 Introduction

★ One is often interested in situations in which only *part* of a quantum system is measured, and one wants to know what this can tell us about the other parts. Measurements of this sort were made famous by the 1935 paper of Einstein, Podolsky, and Rosen, and they arise quite frequently in discussions of quantum information

3.2 Two qubit example

★ Let us look at a simple example, the one shown in Fig. 3, where we assume an initial state $|\psi_0\rangle = |+0\rangle$ at time t_0 leading to

$$|\psi_1\rangle = T(t_1, t_0)|+0\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \tag{11}$$

at time t_1. Suppose the measurement outcome is $D = 1$. What can we say about qubit b just before or just after the measurement takes place?

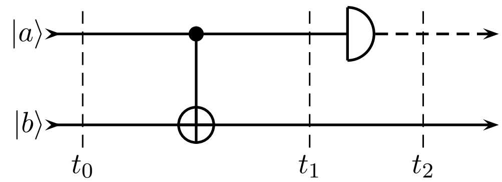

Figure 3: Controlled-not gate followed by detector

• The first and obvious answer is that qubit b is in the state $|1\rangle$. One can come to this conclusion by "looking at" the right side of (11) and noticing that $|a\rangle = |1\rangle$ is clearly paired with $|b\rangle = |1\rangle$.

• Another, perhaps safer, procedure is to introduce the standard orthonormal basis $\{|jk\rangle\}, j = 0, 1; k = 0, 1$ for the two qubits, the product of the standard bases for the individual qubits, at t_1, and use the Born rule to calculate

$$\Pr(00) = 1/2 = \Pr(11), \quad \Pr(01) = 0 = \Pr(10). \tag{12}$$

From this it follows that the conditional probability $\Pr(b{=}1 \,|\, a{=}1) = 1$, consistent with the previous conclusion. Here we are invoking CEP: The outcome $D = 1$ means the a qubit was in the state $[1]$ and not $[0]$ at time t_1.

★ While it seems safe to assign $|1\rangle$ to qubit b at time t_1, one might still be a little sceptical, for the following reason. The expansion on the right side of (11) employs a particular basis, but there are many possible bases for a two qubit system, and if we use the $\{|+\rangle, |-\rangle\}$ basis for qubit b, one can write $|\psi_1\rangle$ in (11) in the form

$$|\psi_1\rangle = \big(|0+\rangle + |0-\rangle + |1+\rangle - |1-\rangle\big)/2. \tag{13}$$

It is obvious from "looking at" this equation—and one can check the result by calculating the probabilities $\Pr(a = 1, b = +)$, etc.—that if qubit a is in the state 1, then qubit b is with probability 1/2 in the state $+$ and with probability 1/2 in the state $-$. (As usual, we leave off the ket symbols and use $+$ for $|+\rangle$ when this will not confuse.) Also, if we measure the b qubit in the $\pm$ basis instead of in the $0, 1$ (standard) basis at the same time as a is measured in the $0, 1$ basis, the results of the measurement will confirm these probabilities. All of this sounds plausible. But of course we cannot say that the b qubit is simultaneously in the state $|1\rangle$ and also at the same time in one of the states $|+\rangle$ or $|-\rangle$, for that is quantum nonsense.

• Isn't this just the same sort of problem we encountered when we assumed a certain initial state $|\psi_0\rangle$ at t_0, and found that the probabilities produced by applying the Born rule depend upon the choice of basis at t_1? Indeed, it is precisely the same sort of problem, and it comes up whenever one calculates probabilities in quantum mechanics. The choice of sample space (orthonormal basis or decomposition of the identity) is crucial, and that choice is not unique.

• If the problem is the same, so is the solution. The state $|\psi_1\rangle$ resulting from the unitary evolution of $|\psi_0\rangle$ can be thought of as a pre-probability, which conveniently generates the probability distribution for whatever basis is of interest. In the same way, if $D = 1$ is the detection outcome for qubit a, we can regard $|b\rangle = |1\rangle$ as the state of particle b in the sense that it is the *correct pre-probability* for calculating a probability for the (physical) state of qubit b in whatever basis interests us, including the basis $\{|+\rangle, |-\rangle\}$.

★ Suppose we are interested in properties of qubit b at a later time t_2, Fig. 3, rather than just at t_1, the time just before the measurement of qubit a took place. What can we say? A full analysis of what can be said about b at t_1 *and* t_2 requires the use of the machinery of quantum histories, with checks of consistency, and that is outside the scope of these notes.

• There are nonetheless some results that can be stated quite simply, even though showing that they are correct must be left as an exercise. Because the time evolution of b is trivial between t_1 and t_2 the same pre-probability used at t_1 can be used at t_2. That is, one can introduce an arbitrary basis $\{|w^+\rangle, |w^-\rangle\}$ at t_2 and calculate the probabilities of $[w^+]$ and $[w^-]$ using $|1\rangle$ as a pre-probability. One cannot, however, choose arbitrary bases at *both* t_1 and t_2. and use $|1\rangle$ as the pre-probability for both.

☐ Exercise. How should the preceding comments be modified in case there is a one qubit unitary U acting on the b qubit between the times t_1 and t_2 in Fig. 3?

3.3 General $\mathcal{H}_a \otimes \mathcal{H}_b$

★ The preceding discussion can be generalized as follows. For a combined system $\mathcal{H}_a \otimes \mathcal{H}_b$, where $\mathcal{H}_a$ and $\mathcal{H}_b$ can have arbitrary (finite) dimensions, let $|\psi_1\rangle$ be the state that evolves unitarily from an initial state $|\psi_0\rangle$, and thus functions as a pre-probability at time t_1. Now suppose we have a detector D for system $\mathcal{H}_a$ which determines whether it is in one of the states of a particular orthonormal basis $\{|a^j\rangle\}$, where $|a^j\rangle$ results in the (macroscopic) outcome $D = j$. To determine the pre-probability for $\mathcal{H}_b$, given some outcome of the detector, we expand $|\psi_1\rangle$ in terms of the $\{|a^j\rangle\}$ basis in the form

$$|\psi_1\rangle = \sum_j |a^j\rangle \otimes |\beta^j\rangle, \tag{14}$$

where the $\{|\beta^j\rangle\}$ are *expansion coefficients* determined uniquely by $|\psi_1\rangle$ and the basis$\{|a^j\rangle\}$, and are not (at least in general) elements of an orthogonal basis.

• It is easy to check, by using the Born rule and making the CEP assumption that $|a^k\rangle$ at t_1 leads to $D = k$ a short time later, that

$$\Pr(D = k) = \langle\beta^k|\beta^k\rangle. \tag{15}$$

★ Next suppose we are interested in a decomposition $\{Q^p\}$ of the identity I_b of $\mathcal{H}_b$. The joint probability distribution of the $\{[a^k] = |a^k\rangle\langle a^k|\}$ and the $\{Q^p\}$ can be computed using Born's rule:

$$\Pr(a^k, Q^p) = \langle\psi_1|\Big([a^k] \otimes Q^p\Big)|\psi_1\rangle = \langle\beta^k|Q^p|\beta^k\rangle. \tag{16}$$

• Consequently the *conditional* probability for Q^p given $D = k$ is

$$\Pr(Q^p \,|\, a^k) = \Pr(a^k, Q^p)/\Pr(a^k) = \langle\beta^k|Q^p|\beta^k\rangle/\langle\beta^k|\beta^k\rangle = \langle\bar\beta^k|Q^p|\bar\beta^k\rangle, \tag{17}$$

where

$$|\bar\beta^k\rangle = |\beta^k\rangle/\sqrt{\langle\beta^k|\beta^k\rangle}. \tag{18}$$

is a normalized version of $|\beta^k\rangle$; $\langle\bar\beta^k|\bar\beta^k\rangle = 1$.

★ Suppose we are interested in knowing the properties of system b at times earlier than or later than t_1, the time at (or just before) which the measurement takes place on system a?

• As long as system b remains isolated between the time of interest, let us denote it by t_1', and t_1, the pre-probability $\langle\bar{\beta}^k|\bar{\beta}^k\rangle = 1$ can be "moved" forwards or backwards in time using the unitary time development operator $T_b(t', t)$ appropriate to system b. See the comments at the end of Sec. 3.2. For the example shown in Fig. 3 one cannot place t_1' earlier than the CNOT gate, as it is only after this gate has acted that system b is isolated.

• The situation is analogous to what one has when applying Born's rule for an isolated system given $|\psi_0\rangle$ at time t_0. Unitary time development can be used to find $|\psi_1\rangle$ at a different time t_1 provided the system remains isolated between t_0 and t_1, and one can use $|\psi_1\rangle$ to calculate probabilities of different properties at t_1 using the Born rule.

3.4 Wave function collapse

★ Observe that we have two equally good ways to compute the conditional probability of the property (projector) Q^p of system b given that the measurement of a has resulted in the outcome $D = j$. First, one can simply compute the joint probability distribution $\Pr(a^j, Q^p)$ using the Born rule, and from it the conditional $\Pr(Q^k \mid a^j)$ using the standard rules of probability theory, the first equality in (17). Or one can "collapse" the wave function $|\psi_1\rangle$ as written out in (14) onto the $|\beta^j\rangle$ that corresponds to the measurement outcome, and then normalize this ket, (18), to obtain the last expression on the right side of (17).

• As long as it is regarded as a calculational tool there can be no objection to the collapse procedure. It is efficient, especially if one is only concerned with one measurement outcome $D = j$ for system a and perhaps a large number of properties Q^p for system b. It also allows one to calculate conditional probabilities for various different decompositions of the identity I_b. In this respect $|\bar{\beta}^j\rangle$ is functioning as a *conditional pre-probability*, in very much the same way as $|\psi_1\rangle$ functions as a pre-probability when applying the Born rule.

• It is, however, worth emphasizing that this "collapse" is *not* any kind of of *physical* effect. A pre-probability is a calculational device, not (at least in general) a description of what is actually going on in a quantum system. A great deal of confusion has arisen from failing to make this distinction. For example, the totally mistaken idea that quantum mechanics contains mysterious long-range influences that violate the principles of relativity theory. Note that the *probability* that a distant object is in some state can change abruptly even though the object itself remains unchanged, because probabilities are a form of partial knowledge about a system, and partial knowledge can (and should) change in the light of new information.

• Wave function "collapse" also arises in discussions of nondestructive measurements, see Sec. 4. The idea is basically the same; once again, it is a calculational tool, not a physical effect.

3.5 Measurements in QCQI

★ QCQI in Secs. 2.2.3 through 2.2.8 contains a rather confusing discussion of measurements, reflecting the confusion found in typical textbook presentations of quantum theory. In particular, destructive measurements, which are the simplest kind, are not properly addressed by postulate 3 in Sec. 2.2.3. And since the treatment is not based on a proper quantum analysis of the measuring apparatus and its interaction with the measured system, the meaning of this postulate is not altogether clear.

• What one finds in QCQI is a *calculational scheme*. When properly applied it gives the correct answers.

4 Nondestructive Measurements

4.1 Introduction

• Von Neumann's discussion of quantum mechanics employed nondestructive measurements, and this has strongly influenced the presentation one finds in textbooks and other books, including QCQI. But in the laboratory nondestructive measurements are more difficult to achieve than the destructive measurements discussed in Sec. 2, and this is one reason why textbook discussions of measurements are not very helpful in understanding situations which are relevant to quantum information and computation.

• Properties of nondestructive measurements can be worked out fairly easily using the principles that apply to partial measurements, as discussed in Sec. 3.

4.2 One qubit example

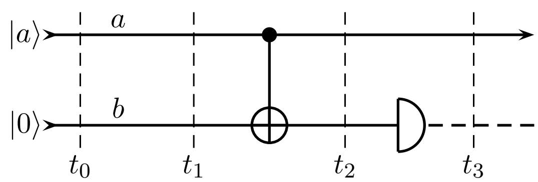

Figure 4: Circuit for nondestructive measurement of qubit a in the standard basis.

★ A simple example showing how one might carry out the nondestructive measurement in the standard basis of a single qubit a is shown in Fig. 4. The circuit employs a second *ancillary* qubit b which is initially in the state $|0\rangle$. The CNOT gate copies the standard basis states of the a qubit onto qubit b, and the latter is then measured using our standard destructive measuring device. Consequently, if $|a\rangle = |0\rangle$ or $|1\rangle$ at t_0, it has the same value at time t_3 after the measurement, and this value corresponds to the (macroscopic) value of $D = 0$ or 1.

★ But suppose that $|a\rangle$ is in some other state, say

$$|a\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle, \quad |\alpha_0|^2 + |\alpha_1|^2 = 1, \tag{19}$$

at t_0, what will be the result of the measurement? Unitary time development of the initial state

$$|\psi_0\rangle = |a\rangle \otimes |0\rangle = \bigl(\alpha_0|0\rangle + \alpha_1|1\rangle\bigr) \otimes |0\rangle, \tag{20}$$

results in

$$|\psi_2\rangle = \alpha_0|00\rangle + \alpha_1|11\rangle \tag{21}$$

at time t_2.

• According to the discussion in Sec. 3, if the detector is in the state $D = 0$ at time t_3, $|a\rangle = |0\rangle$ is the correct pre-probability for qubit a at time t_2, whereas if $D = 1$ we should use $|a\rangle = |1\rangle$. Thus the measurement outcome is correlated with the corresponding property of qubit a at the time t_2, and thus also at a later time t_3 just after the measurement is complete, assuming qubit a remains isolated and its time development operator is trivial (I) for the time interval from t_2 to t_3.

∘ Note that if the ancillary qubit, the CNOT gate and the detector D are all thought of as part of the nondestructive measuring device, it can be shown that at time t_1, just before the nondestructive measurement began, qubit a had the property [0] or [1] corresponding to the later measurement outcome. Consequently, this device also functions as a measurement of a prior property in the same sense as an ordinary destructive measurement.

□ Exercise. Show that introducing the decomposition $\{[0]_a \otimes I_b, [1]_a \otimes I_b,\}$ at t_1 as well as at t_2 leads to a consistent family, and check the correctness of the preceding statement.

• Even though the measurement process shown in Fig 4 is nondestructive in the sense that the states (properties) $|0\rangle$ and $|1\rangle$ for qubit a are unaffected, it is destructive for other properties. For example, if $|a\rangle = |+\rangle$ at t_0, then at t_3 the values $|+\rangle$ and $|-\rangle$ are equally likely, and the same if the initial state is $|a\rangle = |-\rangle$. Thus if we employ the $\{|+\rangle, |-\rangle\}$ basis the final state (in the sense of property) of a is uncorrelated with the initial state. While quantum measurements can be nondestructive for certain properties, they tend at the same time to destroy or at least perturb other properties.

4.3 General case

★ One can easily generalize the simple nondestructive measurement of Fig 4 to the case of many qubits, or to a system $\mathcal{H}_a$ with a Hilbert space of any (finite) dimension, by introducing appropriate ancillary apparatus, and treating $\mathcal{H}_a$ and the apparatus as a single quantum system. For the formalism, see Sec. 18.5 of CQT. That is, for a *fixed* basis $\{|a^j\rangle\}$ of $\mathcal{H}_a$, one can arrange things so that if $\mathcal{H}_a$ starts off in one of the basis states, say $|a^k\rangle$, at t_0, it is still in this state at the end of the measurement, and the detector is in the macroscopic state $D = k$. If the state of $\mathcal{H}_a$ just before the measurement takes place is a linear combination,

$$|\psi_0\rangle = \sum_j c_j |a^j\rangle, \tag{22}$$

then the outcome $D = k$ will occur with probability $|c_k|^2$, and if $D = k$ is the outcome that actually occurs, the state of the particle (in the sense of a pre-probability) after the measurement will be $|a^k\rangle$.

• One is often interested in situations in which a nondestructive measurement of this sort is applied to only part of a quantum system; e.g., the system $\mathcal{H}_a$ might be in an entangled state with another system $\mathcal{H}_c$. If the combined system is in a state $|\Psi_0\rangle$ just before the measurement, and the measurement takes place in such a short time that neither $\mathcal{H}_a$ nor $\mathcal{H}_c$ change significantly (apart from the measuring process itself), then the state of $\mathcal{H}_a \otimes \mathcal{H}_c$ after the measurement, given that the outcome was $D = k$, is

$$\Big([a^k] \otimes I_c\Big)|\Psi_0\rangle \quad \text{or} \quad \frac{\Big([a^k] \otimes I_c\Big)|\Psi_0\rangle}{\sqrt{\langle\Psi_0|\Big([a^k] \otimes I_c\Big)|\Psi_0\rangle}}, \tag{23}$$

where $[a^k] = |a^k\rangle\langle a^k|$ is the projector on $\mathcal{H}_a$ corresponding to the measurement outcome, and the second expression is the normalized form of the first. Once again, "state" is to be understood as a pre-probability. The result (23) is an example of the "projection postulate", or "Lüders' rule."

∘ Nowadays such postulates are no longer needed, as they are consequences of general rules for assigning probabilities to consistent histories.