Dense Coding, Teleportation, No Cloning

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References:
NLQI = R. B. Griffiths, "Nature and location of quantum information" Phys. Rev. A 66 (2002) 012311, also
http://arxiv.org/archive/quant-ph/0203058

QCQI = *Quantum Computation and Quantum Information* by Nielsen and Chuang (Cambridge, 2000). Look up "superdense coding", "teleportation", "no-cloning" in the index. Add p. 187 to the teleportation references.

Contents

1 Fully Entangled States

• As previously noted, entangled states on a tensor product are peculiarly quantum in the sense that there is no good classical analog for them. Dense coding and teleportation are two processes which make use of entangled states, and for this reason appear somewhat strange from an everyday "classical" perspective.

★ We shall later introduce a measure of entanglement for pure states, but for the moment all we need are *fully entangled* (or *maximally entangled*) states. Let $\mathcal{H}_a$ and $\mathcal{H}_b$ be two Hilbert spaces of the same dimension d. Any state on $|\Psi\rangle$ on $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ can be written in the Schmidt form:

$$|\psi\rangle = \sum_j \lambda_j |a_j\rangle \otimes |b_j\rangle, \tag{1}$$

where $\{|a_j\rangle\}$ and $\{|b_k\rangle\}$ are suitable orthonormal bases (which depend upon $|\psi\rangle$).

• A fully entangled state is one for which all the λ are equal (or equal in magnitude if one does not impose the condition $\lambda_j > 0$), and thus equal to $1/\sqrt{d}$ if $|\psi\rangle$ is normalized.

• In the case of two qubits, $d = 2$, the Bell states

$$\begin{aligned}
|B_0\rangle &= \big(|00\rangle + |11\rangle\big)/\sqrt{2}, \\
|B_1\rangle &= \big(|01\rangle + |10\rangle\big)/\sqrt{2}, \\
|B_2\rangle &= \big(|00\rangle - |11\rangle\big)/\sqrt{2}, \\
|B_3\rangle &= \big(|01\rangle - |10\rangle\big)/\sqrt{2},
\end{aligned} \tag{2}$$

are examples of fully-entangled states which form an orthonormal basis.

★ Fully entangled states can also be characterized in the following way. Let the reduced density operators for a normalized $|\psi\rangle$ on $\mathcal{H}_a$ amd $\mathcal{H}_b$ be defined in the usual way:

$$\rho_a = \mathrm{Tr}_b([\psi]), \quad \rho_b = \mathrm{Tr}_a([\psi]). \tag{3}$$

For a fully entangled state,

$$\rho_a = I/d = \rho_b. \tag{4}$$

∘ Note that we are assuming that the reduced density matrices come from a pure state $|\psi\rangle$ on $\mathcal{H}_a \otimes \mathcal{H}_b$, and not from a mixed state represented by a density operator. (Entanglement for mixed states is a complex problem which is far from well understood at the present time.)

□ Exercise. Show that only one of the equalities in (4) is actually needed, as the second is a consequence of the first (and vice versa).

★ Fully entangled states do not have a unique Schmidt decomposition. Given *any* orthonormal basis $\{|a_j\rangle\}$ of $\mathcal{H}_a$, there is an orthormal basis $\{|b_k\rangle\}$ of $\mathcal{H}_b$, one that depends both on $\{|a_j\rangle\}$ and on $|\psi\rangle$, such that (1) holds (with $\lambda_j = 1/\sqrt{d}$).

□ Exercise. Prove this assertion by expanding $|\psi\rangle$ in the form $\sum_j |a_j\rangle \otimes |\beta^j\rangle$, and using (4).

★ Given two normalized fully entangled states $|\psi\rangle$ and $|\phi\rangle$ on $\mathcal{H}_a \otimes \mathcal{H}_b$ one can always find unitary operators U and V on $\mathcal{H}_a$ and $\mathcal{H}_b$ such that

$$|\phi\rangle = (U \otimes V)|\psi\rangle. \tag{5}$$

• That there is *some* unitary operator W on $\mathcal{H}_a \otimes \mathcal{H}_b$ mapping $|\psi\rangle$ to $|\phi\rangle$ is a consequence of the fact that they have the same norm. What is special about (5) is that W is of the form $U \otimes V$. It will be convenient to refer to call such an operator a *local unitary*. The idea of "local" is that one thinks of the subsystems $\mathcal{H}_a$ and $\mathcal{H}_b$ as located in two separate laboratories where Alice applies U to the first and Bob applies V to the second system.

□ Exercise. Another class of pure states mapped into each other by local unitaries are the (normalized) product states. Can you think of other classes? What is the most general class? [Hint: Schmidt.]

★ For any $d \geq 2$ one can find an orthonormal basis for $\mathcal{H}_a \otimes \mathcal{H}_b$ consisting of fully entangled states, analogous to the Bell basis in (2). These bases are not unique, there are always many possibilities.

2 Dense Coding

★ The phenomenon of "dense coding" is based on the observation that given some state belonging to the Bell basis (2) there are local unitaries on $\mathcal{H}_a$ which will map it onto any of the other states belonging to the basis, apart from an overall phase. There are similar unitaries on $\mathcal{H}_b$.

∘ Thus if we start with $|B_0\rangle$, it is mapped to $|B_1\rangle$ by $X \otimes I$, to $|B_2\rangle$ by $Z \otimes I$, and to $|B_3\rangle$ (up to a phase) by $Y \otimes I$.

• As a consequence, given an entangled state $|B_j\rangle$ shared between Alice's and Bob's laboratories, either one of them can convert it into another basis state $|B_k\rangle$ by applying an appropriate unitary.

• On the other hand, neither of them can determine by local measurements, i.e., separate measurements on $\mathcal{H}_a$ and $\mathcal{H}_b$, which $|B_j\rangle$ they jointly possess. If they both carry out measurements and compare them using a classical channel (telephone) they can make some distinctions. For example, measurements in the standard basis when compared with each other will distinguish $|B_0\rangle$ from $|B_1\rangle$ and $|B_3\rangle$, but not from $|B_2\rangle$. To carry out a measurement in the Bell basis requires either bringing the qubits together, or else doing something like teleportation, which requires another qubit pair in a *known* Bell (or other fully entangled) state.

• A convenient way of imagining how one of the $|B_j\rangle$ might be shared by Alice and Bob is to suppose that Bob produces it in some $\mathcal{H}_a \otimes \mathcal{H}_b$ system in his laboratory (e.g., by photon down conversion), and then ships the $\mathcal{H}_a$ part to Alice over a perfect quantum channel. Putting $\mathcal{H}_a$ inside a carefully constructed box so that it will not be disturbed, and sending the box to Alice by parcel service, is to be thought of as a quantum channel — though not a very practical one, given current technology. It is more realistic to imagine an optical fiber between the laboratories, and sending one of two down-converted photons through the fiber.

★ The protocol for dense coding is illustrated in the following figure, which shows a quantum circuit with parts in Alice's (upstairs) and Bob's (basement) laboratories.

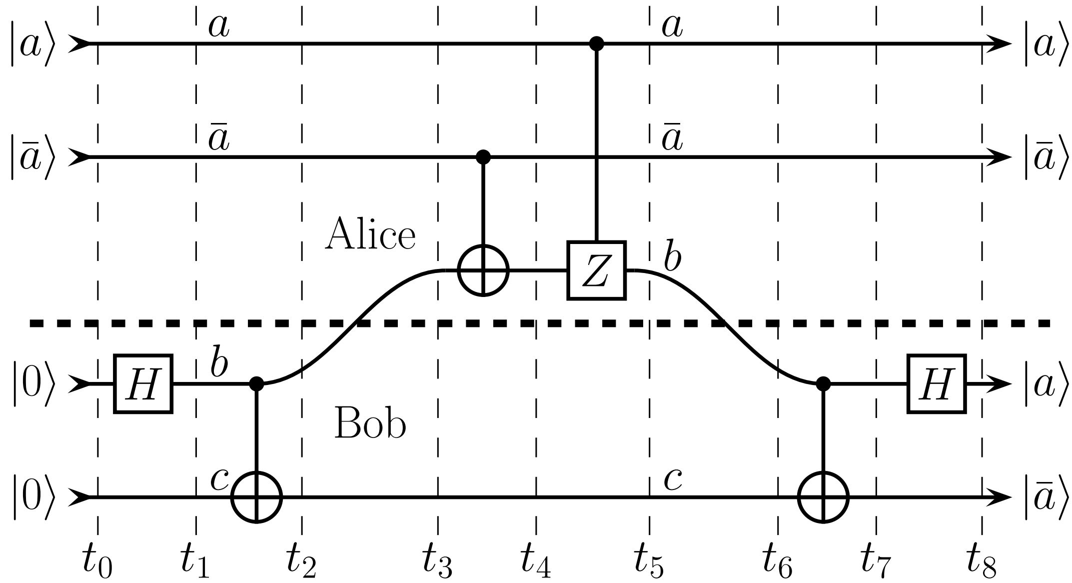

Figure 1: Circuit for dense coding.

• The part of the circuit preceding t_2 is simply a device to turn the product state $|00\rangle$ of two qubits b and c in Bob's laboratory into the entangled Bell state $|B_0\rangle$. The b qubit is then shipped upstairs to Alice. (Alternatively, Alice could produce the entangled bc pair in her labortory and ship c downstairs to Bob.)

• Alice then carries out one of four unitary operations on qubit b which either leave the bc combination in $|B_0\rangle$ or map it into one of the $|B_j\rangle$ with $j > 0$. Whereas one could imagine Alice doing this by hand, the figure shows a quantum circuit in which the unitary is controlled by two qubits a and $\bar{a}$, which are initially in one of the four standard basis states $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ corresponding to the four possible messages which Alice can transmit to Bob by this means.

• At t_5 qubit b is shipped back to Bob. The circuit in his lab following t_6 is just the mirror image of the one used to produce $|B_0\rangle$ in the first place, and its purpose is to measure which Bell state the bc pair is in. The result of the measurement will be two qubits whose states, as indicated in the diagram, are the same as Alice's input, assuming always that both $|a\rangle$ and $|\bar{a}\rangle$ are initially either $|0\rangle$ or $|1\rangle$.

□ Exercise. What happens if one chooses initial $|a\rangle = |x^+\rangle = |+\rangle$ and initial $|\bar{a}\rangle = |0\rangle$? What state results at t_8 from unitary time evolution? Suppose at this time the four qubits are measured in the standard basis; what will one find?

★ What makes this process seem paradoxical, and gives rise to the name "dense coding", is the fact that *only one* qubit, b, passes from Alice's laboratory to Bob's during the time interval between Alice's preparation and Bob's measurement. On the other hand, *two* bits of information needed to identify one out of four possible messages have somehow passed between them. To put it another way, if there were only a one bit classical channel between Alice and Bob, it would have to be used twice to get the message through, whereas the quantum channel is only used once, at least if we ignore what happened before t_3.

◦ In summary, it looks as if a *single* qubit can carry *two bits* of classical information!

• But closer inspection shows that things are not quite so simple. Let us imagine, for example, that qubit b was captured by some outsider (traditionally know as an eavesdropper, or Eve) who wanted to listen in on the message between Alice and Bob. What could she learn about the values of a and $\bar{a}$ from measuring or carrying out other operations on b? Absolutely nothing. From this point of view one might argue that rather than two bits of information, qubit b contains no information (of the relevant sort) at all! In the same way one can show that qubit c by itself contains no information about a and $\bar{a}$ before t_6 when the measurement occurs. Thus in some sense it is only the *combined* system of b along with c that "contains" or "carries" the information; one can say that the information resides in *correlations* between these qubits.

□ Exercise. Show that no information is present in either qubit b or qubit c separately at the time t_5, by computing the reduced density operator of each qubit and showing that it is independent of a and $\bar{a}$.

★ Some insight into what is going on is provided by the following analogy. Imagine that the quantum channel between Alice and Bob is replaced by a classical channel, which I like to think of as a pipe which can carry colored slips of paper. In the basement Bob has a machine which puts out two slips of paper of the same color, red(R) or green (G) in a random fashion, such that RR or GG is produced with probability 1/2. The first slip is sent upstairs to Alice, who sends a message back to Bob in the following fashion. For 0, she returns a slip with the same color, and for 1 she returns a slip with the opposite color (G if she received R, R if she received G). Bob reads the message by comparing the color of the slip in his possession with the one sent back by Alice: if they are the same that signifies 0, if they are different that means 1.

• The main point of this analogy is that it shows how the message is stored in correlations rather than in individual slips of paper. If Eve steals the slip which Alice is sending back to Bob, she will learn nothing about the message, for the probability that it will be red or green is 1/2, independent of whatever message Alice decided to send.

◦ There is actually a closer connection between this analogy and the quantum circuit in Fig. 1 than one might at first suppose; there is a way of describing the quantum process using a particular consistent family of histories which makes it look "almost" like the classical situation just described. (Details are given in Sec. V of NLQI.)

★ However, in the classical case Alice can only send one bit of information per slip of paper, whereas in the quantum case she can send two bits per qubit, so there is still a quantum mystery.

• One way of viewing the mystery is to recall that, Sec. 1, the Bell basis has the property that given an initial state is one of the Bell basis states, Alice herself, with no help from Bob, can transform it to *any one* of the other three basis states.

◦ Contrast this with what happens when one uses a basis of product states such as the standard basis $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$. Here it is obvious that there is no way by which Alice by herself can change $|00\rangle$ (for example) into each of the other basis states. She can only reach one other basis state, not all three.

• Consequently, Alice can put two bits of information into the two qubit system bc if it starts off in a fully-entangled state, but only one bit of information if it starts off in a product state. The latter is what corresponds best with our "classical" experience of the world, and that is one reason we find quantum entanglement peculiar and perplexing.

★ Dense coding can be generalized in an obvious way to systems with $d > 2$. A total of d^2 different messages can be sent through a d-dimensional quantum channel, provided Alice and Bob share a fully-entangled state to begin with.

3 Teleportation

★ Teleportation resembles dense coding in that it requires the presence of a fully entangled state at the outset. However, its goal is basically different: rather than send a larger-than-expected amount of classical information over a quantum channel, the idea is to send a quantum state over a classical channel.

• Figure 10.2 shows the basic arrangement for teleportation in the form of a circuit. As in Fig. 1, the elements preceding t_2 create an entangled state $|B_0\rangle$ on qubits b and c, and the b qubit is sent to Alice's lab over a quantum channel. Next Carol brings to Alice a state $|\psi\rangle$ to be teleported to Carol's associate Charlie, who is waiting in Bob's laboratory.

• The circuit in Alice's laboratory between t_3 and t_5 followed by the standard-basis detectors D^a and D^b serves to measure the system consisting of qubits a and b in the Bell basis. Note that this is basically the same arrangement as one finds in Fig. 1 following t_6 (where the detectors are not shown) and its purpose is the same.

• The outcomes of the measurements are "classical" signals, shown as heavy lines, which can be transmitted to Bob over a classical channel, for example, an ordinary phone line. At this point Bob carries out a unitary transformation on qubit c, the nature of which is determined by the message received from Alice. The figure shows an automated version: if

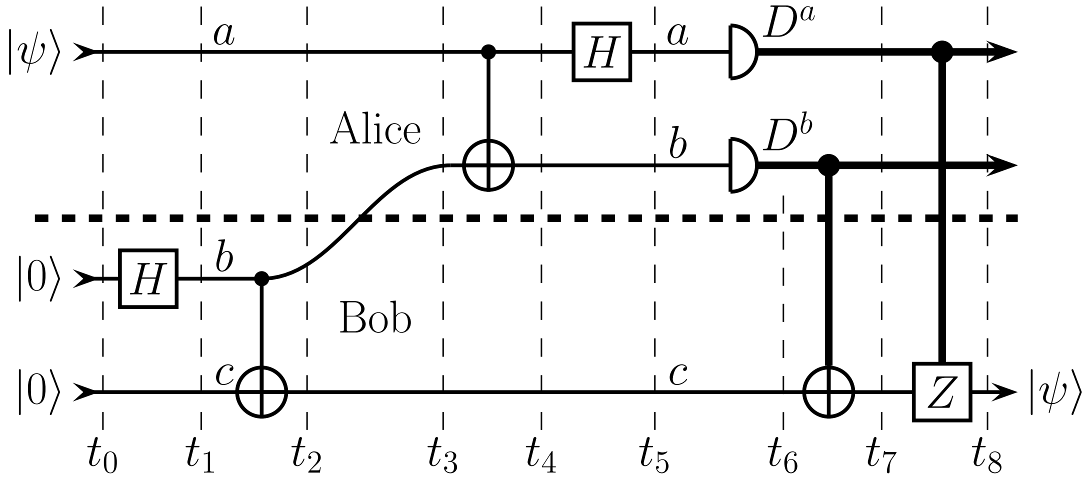

Figure 2: Teleportation.

$D^b = 1$, an X or "not" operation is performed on c, and if $D^a = 1$ a Z or "phase" operation is performed on c, whereas if D^a or D^b is 0 the corresponding operation is not carried out.

★ The net result is that after these operations qubit c, which Bob hands to Charlie, is in the same state $|\psi\rangle$ as the qubit a which Carol brought to Alice. This is the sense in which the quantum state $|\psi\rangle$ has been *teleported* with the help of two classical bits of information, the outcomes of the D^a and D^b measurements, and without the need for a "real" quantum channel from Alice to Bob.

◦ Things are, of course, not quite that simple, for a quantum channel was needed to set up the entangled bc state essential for teleportation. However, this was used *earlier*, possibly even before Carol created the state $|\psi\rangle$ to bring to Alice. One can well imagine that even if a good quantum channel were available to Bob and Alice, it might be advantageous to employ it to produce a collection of pairs of entangled qubits on weekends when the rates are low, and then teleport using a classical channel during the week. Or, teleportation might be reserved for use in an emergency when one needs to transmit a quantum state, but the quantum channel has broken down. (To be sure, using current technology it is not possible to preserve a pair of qubits in a fully entangled state for any significant length of time, but we must leave a few problems to future generations of engineers.)

• Carol and Charlie can check whether the claim of Alice and Bob to be operating a good teleporting service is correct. Carol creates various different 1-qubit states which she brings to Alice, and Charlie measures the states received from Bob in the $|\psi\rangle$, $|\tilde{\psi}\rangle$ basis, where $|\tilde{\psi}\rangle$ is the state orthogonal to $|\psi\rangle$. The outcome of one successful measurement could be an accident, but if things work in, say, 20 cases, this is some evidence that teleportation is taking place.

◦ Teleportation Ltd., the firm that employs Alice and Bob, will not guarantee a *perfect* quantum channel, but instead promises to achieve a certain fidelity F or error rate ϵ, such as $\epsilon < 0.01$, or $F > 0.99$.

★ Alice and Bob both know the outcomes of the Bell-state measurements, i.e., the values of D^a and D^b in any particular case in which teleportation is employed, and one might suppose that this would provide them some information about the state $|\psi\rangle$ supplied by Carol. However, this is not at all the case: for a given $|\psi\rangle$, the D^a and D^b outcomes are completely random: each of the four possibilities for (D^a, D^b) occurs with a probability of 1/4.

• This is an example of a very general principle in quantum information which goes under the name of *no cloning* or *no copying*, see Sec. 4. The basic idea is that a quantum channel cannot be "split" into two or more channels without introducing some noise. Consequently, if the channel from a at t_1 to c at t_8 in Fig. 2 is a perfect channel, it is not possible for information about $|\psi\rangle$ at t_8 to be available anywhere except in qubit c itself. The outcome of the D^a and D^b measurements could be in Alice's notebook (for example), and therefore cannot possibly tell one anything about $|\psi\rangle$.

★ In understanding how the circuit in Fig. 2 functions, it can be helpful to replace it with the one shown in Fig. 3, which has no "classical" elements up to time t_8. By consistent histories arguments, or by the "principle of deferred measurements" in QCQI (p. 186), one can show that the final results (D^a, D^b, and $|\psi\rangle$) are the same.

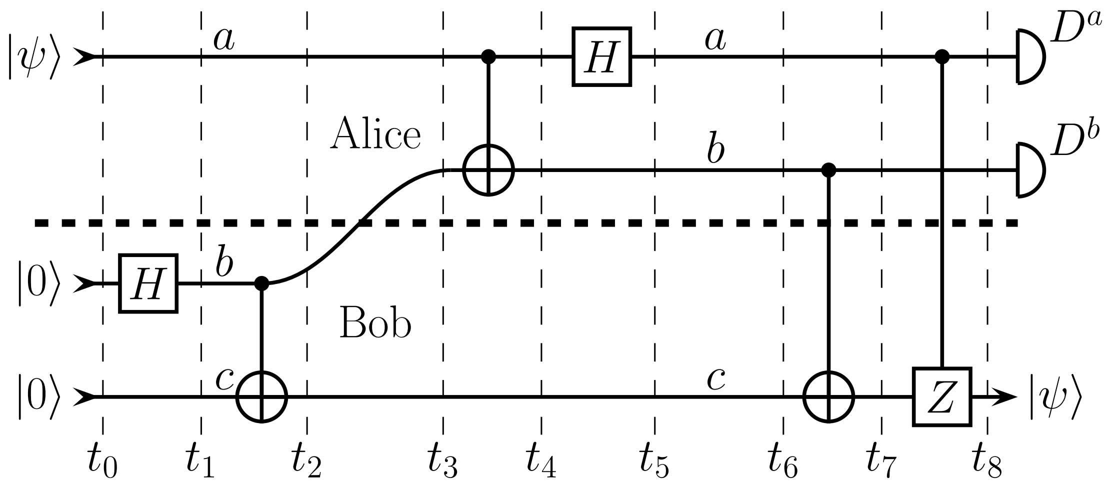

Figure 3: Alternative teleportation circuit.

• Here detection takes place after t_8, and so far as "teleportation" is concerned, nothing would change if there were no detectors: the outcomes obviously do not influence qubit c.

∘ However, carrying out the two operations between t_6 and t_8 requires appropriate "connections" between Alice's and Bob's laboratory, and these, in turn, necessitate using a quantum channel. This undermines the original idea of teleportation, which was to construct a quantum channel without having to use a quantum channel! We introduce the circuit in Fig. 3 not as a "practical" replacement for that in Fig. 2, but as an aid to understanding.

□ Exercise. Assume a general initial state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, and work out the unitary time development of the three qubits in Fig. 3, starting with $|\Psi_2\rangle = |\psi\rangle \otimes |B_0\rangle$ at t_2. In particular obtain $|\Psi_5\rangle$ and $|\Psi_8\rangle$ at times t_5 and t_8.

a) Show that $|\Psi_8\rangle$ is a product state. Then argue that it *has to be* of the form $|ab\rangle \otimes |c\rangle$ if the c output is to be the same as the a input. Can you see from $|\Psi_8\rangle$ why the outcomes of D^a and D^b provide no information about $|\psi\rangle$?

b) Expand $|\Psi_5\rangle$ in the form $\sum_{j,k} |j\rangle \otimes |k\rangle \otimes |c_{jk}\rangle$, i.e., find the kets $|c_{jk}\rangle$. Use these to explain why the circuit in Fig. 2 is successful.

c) Show that at time t_6 in Fig. 3 the information about $|\psi\rangle$ is contained in qubits a and b in the sense that it could be recovered from them by modifying the circuit at later times. (This is in contrast with the situation at t_6 in the circuit in Fig. 2, where information about $|\psi\rangle$ is *not* present in the classical bits representing the measurement outcomes.)

★ If the two "classical" bits representing the measurement outcomes in Fig. 2 contain no information about $|\psi\rangle$, where is that information at, say, t_6? Is it in qubit c? But how can that be, given that the results of the measurements on a and b have yet to reach Bob's laboratory?

• See NLQI Sec. VII for a discussion of these questions. The brief answer is that the information is present in *correlations* between the D^a and D^b outcomes and the qubit c, the nature of which will be evident if you did part (b) of the preceding exercise. There is nothing particularly weird or obscure going on — in particular, there are no magical long-range influences at work — for one can construct a classical analogy for such correlations, as explained in NLQI.

★ In what sense is *quantum* information, in contrast to classical information, teleported?

• If one were simply concerned with the question of whether a qubit is in the state $|\psi\rangle$ rather than the orthogonal $|\tilde{\psi}\rangle$, this type of information can be transmitted in a classical fashion by the simple device of measuring the qubit in the $|\psi\rangle$, $|\tilde{\psi}\rangle$ basis, and sending the information over a classical channel.

• What the teleportation circuit allows one to do is to transmit an unknown quantum state without actually determining what it is.

• Think of the simple perfect quantum channel produced by sending a spin-half particle through a good pipe, or a polarized photon through a good optical fiber. What happens is that whatever goes in comes out again, independent of what it is that goes in.

• In some sense the issue is not "quantum information", but the ability to use *one* circuit or piece of apparatus to transmit the "classical information" distinguishing $|\psi\rangle$ from $|\tilde{\psi}\rangle$ for *many different* possible choices of $|\psi\rangle$, when the choice is not specified in advance. That is what distinguishes a quantum channel from its classical counterpart.

4 No Cloning

★ There is a sense in which quantum information cannot be perfectly copied or "cloned", and this is expressed in various no-cloning theorems.

• The simplest example is illustrated in the following figure, where $\mathcal{H}_a$ and $\mathcal{H}_b$ have the same dimension, but are otherwise arbitrary.

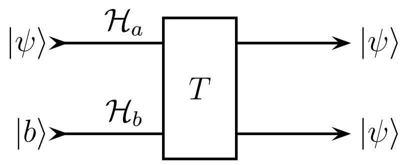

Figure 4: Hypothetical cloning machine.

• Suppose that T and $|b\rangle$ are fixed, while $|\psi\rangle$ can be varied. For a *particular* $|\psi\rangle$ one can always find a unitary T such that $T\big(|\psi\rangle \otimes |b\rangle\big)$ is equal to $|\psi\rangle \otimes |\psi\rangle$. But there is no fixed T that will accomplish this for all possible inputs.

• The argument is straightforward. Suppose the copying circuit can be made to work for two *nonorthogonal* normalized states $|\psi'\rangle$ and $|\psi''\rangle$, and assume that $|b\rangle$ is normalized. Then we have

$$\begin{aligned} T\big(|\psi'\rangle \otimes |b\rangle\big) &= e^{i\phi'}|\psi'\rangle \otimes |\psi'\rangle, \\ T\big(|\psi''\rangle \otimes |b\rangle\big) &= e^{i\phi''}|\psi''\rangle \otimes |\psi''\rangle, \end{aligned} \tag{6}$$

where we have allowed for phases ϕ' and ϕ'', which do not affect the physical interpretation of the final states. Take the inner product of the first of these equations with the second, and use the fact that $T^\dagger T = I$. The result is

$$\langle\psi'|\psi''\rangle = e^{i(\phi''-\phi')}\big(\langle\psi'|\psi''\rangle\big)^2, \tag{7}$$

and upon taking the absolute value of both sides,

$$|\langle\psi'|\psi''\rangle| = |\langle\psi'|\psi''\rangle|^2. \tag{8}$$

• This last equation has two solutions: either $|\langle\psi'|\psi''\rangle|$ is equal to 0 or it is equal to 1. In the first case the two states are orthogonal, and in the second, since we have assumed that both of them are normalized, they are identical apart from a phase factor. (See the exercise following (12).) Both are excluded by our definition of the term "nonorthogonal."

• Conclusion: There is no quantum copying machine that can make two perfect copies (or one perfect copy plus a remaining perfect original) of *two (or more) nonorthogonal states.*

◦ Notice the qualification. There is no rule against making as many perfect copies as one wants of *mutually orthogonal* states using a quantum copying machine. Since any two macroscopically distinct states of the world correspond to orthogonal quantum states, there is no similar restriction on copying "classical objects. Quantum physics is no threat to ordinary photocopying.

★ A second no-cloning result can be obtained by a slight modification of the above argument. Once again, imagine a quantum system with time development given by a unitary operator T acting on a tensor product space $\mathcal{H}_a \otimes \mathcal{H}_c$, where $\mathcal{H}_a$ is of dimension 2 (one qubit), whereas $\mathcal{H}_c$ is of arbitary size, and suppose that

$$|\Psi'\rangle := T\big(|\psi'\rangle \otimes |c\rangle\big) = |\psi'\rangle \otimes |c'\rangle, \tag{9}$$

that is, the result is a perfect copy or, if one prefers, a perfect preservation of the initial $\mathcal{H}_a$ state $|\psi'\rangle$. The states $|\psi'\rangle$ and $|c\rangle$ are assumed to be normalized, and since T is unitary, $|c'\rangle$ must also be normalized, but otherwise we know nothing about it. Unlike (6), there is no need to introduce a phase factor on the right side of this equation, as it can always be incorporated into the definition of $|c'\rangle$.

• Note that in order to produce a *perfect* copy (or preservation) on $\mathcal{H}_a$, $|\Psi'\rangle$ in (9) must be a product state on $\mathcal{H}_a \otimes \mathcal{H}_c$, not an entangled state. If we have an entangled state, then the probability of obtaining $|\psi'\rangle$ at the later time cannot be 1.

☐ Exercise. Justify the preceding statement. [One approach: Form an orthonormal basis $\{|a_j\rangle\}$ of $\mathcal{H}_a$, with $|\psi'\rangle = |a_0\rangle$. Expand $|\Psi'\rangle$ in the $\{|a_j\rangle\}$ with coefficients in $\mathcal{E}$. Use this to calculate the probability of $|\psi'\rangle$.]

• Next suppose there is a second state $|\psi''\rangle$ nonorthogonal to $|\psi'\rangle$, which is also perfectly copied (or preserved):

$$T\big(|\psi''\rangle \otimes |c\rangle\big) = |\psi''\rangle \otimes |c''\rangle. \tag{10}$$

Take the inner product of (9) with (10). The result is

$$\langle\psi'|\psi''\rangle = \langle\psi'|\psi''\rangle\langle c'|c''\rangle. \tag{11}$$

Since $|\psi'\rangle$ and $|\psi''\rangle$ are nonorthogonal, $\langle\psi'|\psi''\rangle$ cannot be 0, so (11) tells us that $\langle c'|c''\rangle = 1$. Since both $|c'\rangle$ and $|c''\rangle$ are normalized, this means that

$$|c'\rangle = |c''\rangle. \tag{12}$$

☐ Exercise. Complete the argument that for normalized states, $\langle c'|c''\rangle = 1$ implies that they are identical. [Hint: What is the norm of $|c'\rangle - |c''\rangle$?] Also show that if $|\langle c'|c''\rangle| = 1$, $|c''\rangle$ is $|c'\rangle$ multiplied by some phase factor $e^{i\phi}$.

★ Since T is a linear operator, (12) combined with (9) and (10) tells us that

$$T\big(|\psi\rangle \otimes |c\rangle\big) = |\psi\rangle \otimes |c'\rangle \tag{13}$$

for *any* $|\psi\rangle$ which is a linear combination of $|\psi'\rangle$ and $|\psi''\rangle$. Because $|\psi'\rangle$ and $|\psi''\rangle$ are nonorthogonal and thus not multiples of each other, their linear combinations form a two-dimensional Hilbert space, in effect the input of a one-qubit quantum channel. What (13) tells us is that this is a perfect channel, and that no information distinguishing different $|\psi\rangle$ in the channel resides in the $|\psi\rangle$-independent state $|c'\rangle$ (and thus in $\mathcal{H}_c$).

• It may not be obvious at first how this result applies to the teleportation circuit in Fig. 3, since the input and output of the channel are different: the a qubit and the c qubit, respectively. All one needs to do is to add at the end of the circuit a unitary operation which exchanges the a and the c qubits, so the output is also the a qubit, and let T in (13) be the unitary corresponding to this augmented circuit.

★ The result can be summarized as follows: if a one-qubit channel is perfect for two nonorthogonal states, it is perfect for all states, and no information distinguishing these states can leak out of it into the environment.

★ In the same way, to check whether a $d > 2$ channel is perfect, it suffices to show that it is perfect for d nonorthogonal states which together span the d-dimensional Hilbert space of the channel. No information can leak out of such a perfect channel.

☐ Exercise. What is meant by "d *nonorthogonal* states"? Suppose we label them as $|\phi^k\rangle$, $1 \le k \le d$. Is it necessary that $\langle\phi^k|\phi^j\rangle$ be nonzero for all j and k, or will the argument go through with some of these inner products equal to 0?