33-658, 758 Quantum Computation and Information Spring Semester, 2014
Assignment No. 11 Due Tuesday, April 8

READING:
GSGC = "Graph States and Graph Codes" on course web page
QCKOP = "Quantum Channels, Kraus Operators, POVMs" on course web page
QCQI = Nielsen and Chuang, Quantum Computation and Quantum Information
QEC = "Quantum Error Correction" on course web page

POVMs: QCQI Sec. 2.2.6; QCKOP Sec. 5
Quantum error correction: QCQI Secs. 10.1, 10.2, 10.3; QEC; E. Knill, R. Laflamme, "Theory of quantum error-correcting codes," Phys. Rev. A 55 (1997) 900; arXiv:quant-ph/9604034
Graph States and Codes: GSGC

READING AHEAD:

EXERCISES:

1. Prepare a one-page description of your term paper. Include the title, some description of the contents, and an indication of the resources you plan to use. E.g., you have located some interesting papers, or you plan to use some of the material in Nielsen and Chuang, or you will be employing their bibliography, or you will be getting advice from Prof. X, etc. If necessary you can change the plan later, but please discuss significant changes with the instructor. You will be contacted promptly if it looks as if there is some problem associated with your proposal.

2. A POVM $\{G_j\}$ for a system s, Hilbert space $\mathcal{H}_s$, can always be thought of as a projective measurement on a larger system consisting of s and an ancilla a, Hilbert space $\mathcal{H}_a$, which is in a known initial state $[0]_a$. The combination, with Hilbert space $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_s$, is subjected to a projective measurement using a (projective) decomposition of the identity $\{\hat{P}_j\}$ chosen in such a way that

$$G_j := \mathrm{Tr}_a\big(\hat{P}_j A_0\big) = \mathrm{Tr}_a\big(A_0 \hat{P}_j A_0\big),$$

where

$$A_0 = [0]_a \otimes I_s$$

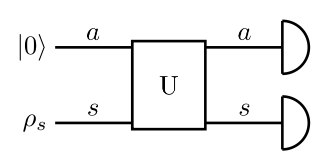
is the projector onto the subspace of $\mathcal{H}$ corresponding to the specified initial state of the ancilla.

a) Why is the second equality in the expression for G_j correct? Note that $\hat{P}_j$ and A_0 do not (in general) commute.

b) Show that the POVM measurement and the projective measurement yield the same probilities for an arbitrary initial state of s, i.e.

$$\mathrm{Tr}_s(\rho_s G_j) = \mathrm{Tr}\Big(\big([0]_a \otimes \rho_s\big)\hat{P}_j\Big).$$

The figure shows a circuit for realizing the projective measurement in a case in which the ancilla a and the system s are both qubits, and at the later time both are measured in the standard basis. The unitary U determines $\{\hat{P}_j\}$ through the relationship



$$\hat{P}_j = U^\dagger P_j U; \quad P_0 = [00],\ P_1 = [01],\ P_2 = [10],\ P_3 = [11].$$

c) Show that if $\hat{P}_j$ is a 4×4 matrix in the standard $|a, s\rangle$ basis in the usual order, $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, then G_j is the upper left 2×2 block of this matrix.

d) Let $\omega = e^{2\pi i/3}$ (third root of 1) and define

$$U = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{6} & \omega/\sqrt{3} & -1/\sqrt{6} & \omega^2/\sqrt{3} \\ 1/\sqrt{6} & \omega^2/\sqrt{3} & -1/\sqrt{6} & \omega/\sqrt{3} \end{pmatrix}$$

Compute the G_j for $0 \leq j \leq 3$ as 2×2 matrices. [Hint. Use the result in (c), and note that each matrix element in $\hat{P}_j$ is a quite simple function of the matrix elements of U.] Check that $\sum_j G_j = I$, and that the G_j are positive operators of rank 1. [Hint. Use trace and determinant.]

e) Optional. If the G_j are multiplied by suitable positive constants they correspond to pure state density operators, so points on the Bloch sphere. Show that these correspond to vertices of a simple geometrical figure.

f) Optional. Design a game for which Alice will do better to use this POVM than to try and measure the qubit s in some orthogonal basis.

3. a) Find a *unitary* circuit based on the idea, see QCQI p. 430, that one can correct a single bit-flip error in their three qubit code by measuring $Z_1 Z_2$ and $Z_2 Z_3$, and using the results to perform appropriate unitary corrections on the code qubits. Use two ancillary qubits, initially in the state $|0\rangle$, to hold the syndrome and control the error correction through appropriate gates. (Five qubits in all: three carrier qubits and two ancillary qubits.) Explain in words how your circuit accomplishes the desired task, and indicate how the syndrome is related to the type of error which occurs.

b) Suppose that instead of a bit flip error, interaction with the environment causes the first qubit in the code to flip to $|0\rangle$ if it is initially $|1\rangle$, but leaves it in the state $|0\rangle$ if it is initially in this state. Will your circuit correct this error? Model interaction with the environment through a unitary transformation

$$|0\rangle|e\rangle \mapsto |0\rangle|e_0\rangle, \quad |1\rangle|e\rangle \mapsto |0\rangle|e_1\rangle,$$

where $|e\rangle$, $|e_0\rangle$, and $|e_1\rangle$ are states in an environment Hilbert space $\mathcal{E}$, and work out how the encoded state $a|000\rangle + b|111\rangle$ along with the rest of the system and the environment evolves unitarily in time. In order to interpret the final state it is helpful to trace out the environment—what is the inner product $\langle e_0|e_1\rangle$?

4. Parts (a) and (b) are from QCQI, p. 441, exercises 10.8 and 10.9.

a) Verify that the three qubit phase flip code

$$|0\rangle_L = |+++\rangle, \quad |1\rangle_L = |---\rangle$$

satisfies the Knill-Laflamme (projector) error correction condition for the set of error operators $\{I, Z_1, Z_2, Z_3\}$.

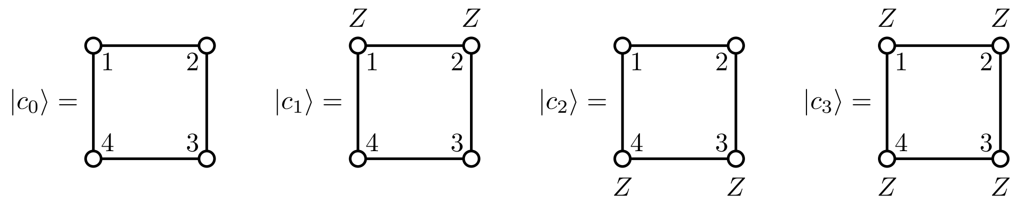
b) Prove that the same code protects against the error set $\{I, P_1, Q_1, P_2, Q_2, P_3, Q_3\}$, where P_i and Q_i are projectors onto the $|0\rangle$ and $|1\rangle$ states, respectively, of the i'th qubit. [Hint. This is easy once you have done (a).]

c) Show that the Knill-Laflamme condition fails if the error operator $Z_1 Z_2$ is added to the set $\{I, Z_1, Z_2, Z_3\}$.

5. The Shor 9 qubit code is not only robust against an arbitrary error on a single qubit, but in addition certain types of two and three qubit errors are correctable, provided these occur on certain specified qubits. For example, two phase flip errors on qubits in the same block. Try and identify as many of these correctable multiqubit errors as you can, giving some indication of how you reached your conclusion.

6. Consider the $n = 4$ square graph code of dimension $K = 4$, where $|c_0\rangle = |G\rangle$ corresponds to the graph in which there are edges between vertices 1 and 2, 2 and 3, 3 and 4, and 1 and 4, and

$$|c_1\rangle = Z_1 Z_2 |c_0\rangle, \quad |c_2\rangle = Z_3 Z_4 |c_0\rangle, \quad |c_3\rangle = Z_1 Z_2 Z_3 Z_4 |c_0\rangle.$$



a) What are the graph basis states obtained when each of the operators X_1, Z_1, $X_1 Z_1$ is applied to the state $|c_1\rangle$? (Always identify the basis state by indicating the collection of qubits where a Z operator has been applied to $|G\rangle$).

b) Show that

$$\langle c_j | Q | c_k \rangle = 0$$

for any operator Q of size 1 (it acts on a single qubit) with zero trace, for $0 \le j, k \le 3$. Use this to show that the distance δ of this code is 2 or more.

c) What does the result in (b) tell you about the reduced density operator ρ_1 on qubit 1 corresponding to a general state

$$|\Psi\rangle = \sum_j \alpha_j |c_j\rangle, \quad \sum_j |\alpha_j|^2 = 1$$

on the coding space? [Hint. If you know $\langle X \rangle_1$, $\langle Y \rangle_1$ and $\langle Z \rangle_1$, what does this tell you about ρ_1?]

d) Find two operators Q and $\bar{Q}$ of size 1 such that

$$\langle c_1 | Q \bar{Q} | c_2 \rangle \neq 0.$$

Explain why this in and of itself means that the distance δ of this code cannot be larger than 2, and thus, given the result in (b), it is $\delta = 2$.