33-658, 758 Quantum Computation and Information Spring Semester, 2014
Assignment No. 9. Due Tuesday, March 25

ANNOUNCEMENT. There will be an hour exam on Tuesday afternoon, April 1, beginning at 3:00 pm (usual class hour). It is closed book, closed notes, no pocket calculators. Bring a pen or pencil.

ANNOUNCEMENT for students taking the 12 unit course. There will *NOT* be a seminar on March 25 (conflicts with the Buhl lecture) or on April 1 because of the hour exam.

READING:
GSGC = "Graph States and Graph Codes" on course web page
PITTENGER = A. O. Pittenger, An Introduction to Quantum Computing Algorithms (Library reserve)
QCKOP = "Quantum Channels, Kraus Operators, POVMs" on course web page
QCQI = Nielsen and Chuang, Quantum Computation and Quantum Information
QEC = "Quantum Error Correction" on course web page

Grover search algorithm: QCQI Sec. 6.1; PITTENGER Sec. 3.3
Quantum channels/operations: QCQI Secs. 8.1, 8.2, 8.3; QCKOP
Error and fidelity measures: QCQI Ch. 9 is a bit dense; try and get the general idea. QCKOP Sec. 3.5.
POVMs: QCQI Sec. 2.2.6; QCKOP Sec. 5

READING AHEAD:
Quantum error correction: QCQI Secs. 10.1, 10.2, 10.3; QEC; E. Knill, R. Laflamme, "Theory of quantum error-correcting codes," Phys. Rev. A 55 (1997) 900; arXiv:quant-ph/9604034
Graph States and Codes: GSGC

EXERCISES:

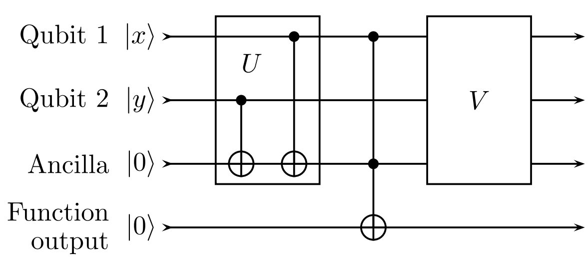
1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course.

1b. Only for students enrolled in 33-758: Summarize in half a page to a page what you learned from the most recent seminar.

2. (Reversible function evaluation and ancilla qubits): Suppose that a quantum algorithm requires a subroutine which computes

$$f(x,y) = \begin{cases} x & \text{if} \quad x \oplus y = 1 \\ 0 & \text{if} \quad x \oplus y = 0 \end{cases}$$

reversibly where x, y are single bits and addition is modulo 2. The most direct method for constructing a circuit to perform $|x\rangle\,|y\rangle\,|0\rangle \mapsto |x\rangle\,|y\rangle\,|f(x,y)\rangle$ requires a single ancillary qubit and results in:

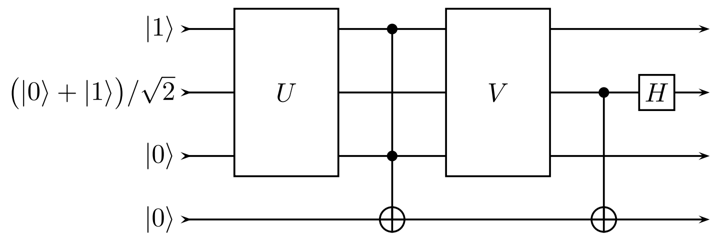


where U is the unitary transformation carried out by the gates inside the rectangular box.

a) Compute the state of the qubit system immediately after U and immediately after the Toffoli gate when the input state to the entire circuit is $|x, y, 0, 0\rangle$ with $x, y \in \{0, 1\}$.

b) Provide a circuit for V which resets the ancilla qubit but leaves qubits 1 and 2 unaltered when the input to the entire circuit is that of part (a). The effect of the entire circuit should then be $|x, y, 0, 0\rangle \to |x, y, 0, f(x,y)\rangle$.

c) Suppose that during the algorithm the function evaluation occurs in the following context:



Determine the final state of the qubit system when V is designed according to the criteria of part (b). Then calculate the probability p_m that a measurement of qubit 2 in the computational basis will yield $m = 0$ or 1.

d) Repeat part (c) for the case where the ancilla resetting operation, V, is omitted. Once again calculate the probability p_m that a measurement in the computational basis will yield $m = 0$ or 1. Do you obtain the same result as in (c)? (This shows why resetting an ancillary qubit to its original value can be important for a quantum computation.)

3. a) Consider Grover's algorithm for one secret, or one marked item, and $N = 8$. Denote the standard basis states by $|0\rangle$, $|1\rangle$, $|2\rangle$, $\ldots|7\rangle$, and suppose the secret is $s = 1$, corresponding to $|1\rangle$. Let G be the Grover operator as defined in QCQI Sec. 6.1.2, and let $|\psi_0\rangle = N^{-1/2}\sum_{x=0}^{N-1}|x\rangle$. Calculate $|\psi_j\rangle = G^j|\psi_0\rangle$ explicitly for $j = 1, 2, \ldots$ until you reach a value at which the probability of a measurement yielding the value $x = 1$ begins to decrease. What is the probability of a measurement yielding the right answer if one uses the optimum number of iterations?

b) Repeat for $N = 16$ and one marked item, but do not calculate the $|\psi_j\rangle$. Instead, use the geometrical construction, QCQI Fig. 6.3, to find the optimum number of iterations and the probability of finding the right answer if a measurement is made upon completing the optimum number of iterations.

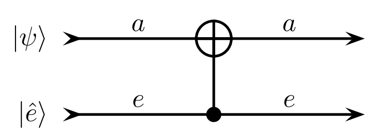
4. Consider the channel for the a qubit, initially in a state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

produced by the following circuit, where the environment e qubit is initially in the state

$$|\hat{e}\rangle = \sqrt{1-p}\,|0\rangle + \sqrt{p}\,|1\rangle,$$

with p between 0 and 1.



a) Work out the Kraus operators (mapping $\mathcal{H}_a$ to $\mathcal{H}_a$) assuming a standard $\{|0\rangle, |1\rangle\}$basis for the e qubit after the interaction, and check that $\sum_l K_l^\dagger K_l = I$. Write the Kraus operators both as 2×2 matrices in the standard basis, and as sums of Pauli matrices.

b) Now find the Kraus operators assuming a basis $\{|+\rangle, |-\rangle\}$ for the e qubit after the interaction, and show that these are linear combinations of the ones you found in (a) with coefficients that form a unitary matrix.

c) Find the channel superoperator $\mathcal{S}$ by finding the matrix S_{jk} such that in terms of Pauli matrices, with $\sigma_0 = I$,

$$\mathcal{S}(\sigma_k) = \sum_j S_{jk}\sigma_j.$$

Check that both sets of Kraus operators, from (a) and (b), give rise to the same superoperator.

d) For $p = 0$, $p = 1/2$, and $p = 1$, describe what the channel does in terms of how it maps the original Bloch sphere by shrinkage along different directions and (possible) rotation.

5. Suppose that a one qubit channel maps $|\psi\rangle$ to a density operator $\rho = \frac{1}{2}(I + \boldsymbol{r}\cdot\boldsymbol{\sigma})$. Find a geometrical interpretation of the fidelity $F(|\psi\rangle)$ in terms of $\mathbf{r}$ and the diameter (line through the center) of the Bloch sphere connecting the point representing $|\psi\rangle$ with its antipode. Also find a formula for F in terms of $\mathbf{r}$ and the point $\mathbf{r}_0$ representing $|\psi\rangle$. Define the fidelity as $\langle\psi|\rho|\psi\rangle$ rather than (as in QCQI) the square root of this quantity. [Hint: You may want to consider a simple example.]