33-658, 758 Quantum Computation and Information Spring Semester, 2014
Assignment No. 5 (Not to be turned in!)

ANNOUNCEMENT. There will be an hour exam on Tuesday afternoon, February 18, beginning at 3:00 pm (usual class hour). It is closed book, closed notes, no pocket calculators. Bring a pen or pencil. There will be no quantum information seminar that afternoon.

The exam will be based on the material in Assignments 1 to 4, and in the lectures up to and including Thursday, Feb. 6. However, the latter part of that lecture dealing with fully entangled states and dense coding is *not* included.

Problems 1 below is the first hour exam in a previous course. The others are practice problems. None are to be turned in. Solutions will be provided on Thursday, Feb. 13.

READING associated with recent and current lectures:
CLRS = Cormen, Leiserson, Rivest, and Stein, Introduction to Algorithms (Library reserve)
DPV = Dasgupta, Papadimitriou, and Vazirani. Algorithms (Library reserve)
QCQI = Nielsen and Chuang, Quantum Computation and Quantum Information
QIT = "Quantum Information Types" Course web site
TELEPORT = "Dense Coding, Teleportation, No Cloning" Course web site

Types of quantum information: QIT. There is also a published paper "Types of Quantum Information" available on the course web site, which is much longer and more detailed. The relevant sections are I, II, and III.

Dense Coding: QCQI Sec. 2.3; TELEPORT; The original paper by Bennett and Wiesner, reference in QCQI p. 119, is also worth reading

No cloning: QCQI Sec. 12.1, and Proposition 12.18 in Sec. 12.6.3; TELEPORT

Teleportation: QCQI Secs. 1.3.7, 4.4; TELEPORT; Bennett et al., Phys. Rev. Lett. 70 (1993) 1895 (Original paper). If you are ambitious, take a look at R. B. Griffiths, Phys. Rev. A 66 (2002) 012311, also in http://arxiv.org/archive/quant-ph/0203058.

READING AHEAD:
Cryptography: Stinson, *Cryptography: theory and practice*, Ch. 1; QCQI Sec. 12.6.1.
Number-theoretic algorithms: CLRS Ch. 31; DPV Ch. 1; QCQI App. 4
Complexity theory: CLRS Ch. 34; DPV Ch. 8; QCQI Ch. 3

EXERCISES (Not to be turned in)

1: From a previous hour exam.

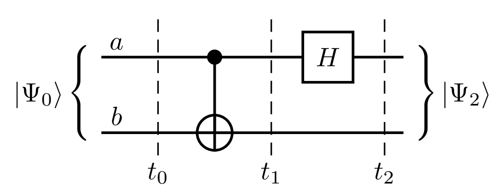
INSTRUCTIONS. This examination consists of one question with 6 parts, each of which is worth between 15 and 20 points. If you cannot make progress on one part, take a look at a later part, for you may be able to do something with it. As on all examinations, *it is important to give a (brief!) indication of your reasoning.* Answer in a way which shows that you *understand* the subject, you really *do* know what you are doing.

There is a short list of formulas on the back side. If you think something else should be there, ask the instructor.

The circuit in the figure results in the state

$$|\Psi_2\rangle = \sqrt{1/3}\,|01\rangle + \sqrt{2/3}\,|10\rangle$$

at time t_2.



a) Find $[0]_a \cdot |\Psi_2\rangle$ and $[1]_a \cdot |\Psi_2\rangle$ (where as usual we think of $[0]_a$ as denoting $[0]_a \otimes I_b$) and use these to calculate the probabilities that qubit a at time t_2 is in the state $|0\rangle$ or $|1\rangle$.

b) Find the reduced density operator ρ_a for qubit a at time t_2. What is its geometrical representation as a point inside the Bloch sphere? (I.e., where in the Bloch sphere is the point located?)

c) Use ρ_a to calculate the same probabilities you calculated in (a), indicating how you use ρ_a to obtain them.

d) Starting from $|\Psi_2\rangle$ use the circuit in the figure to calculate $|\Psi_1\rangle$ and $|\Psi_0\rangle$ at the earlier times t_1 and t_0.

e) Consider a family of histories for the times $t_0 < t_1 < t_2$ of the form

$$[\Psi_0] \odot \{P, \tilde{P}\} \odot \{Q, \tilde{Q}\},$$

where $\tilde{P} = I - P$ and $\tilde{Q} = I - Q$, thus four histories, all with the same initial state $[\Psi_0]$. Explain why this family is consistent if $P = [\Psi_1]$, whatever the choice of the projector Q. You do not have to (and should not) make any reference to the specific $|\Psi_1\rangle$ you computed in (d).

f) Show that the family in (e) is consistent if $P = [+]_a$ and $Q = [0]_a$. ($|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$.)

2. a) Find the normalized kets, expressed as linear combinations of $|0\rangle$ and $|1\rangle$, which in the Bloch sphere (spin half) picture correspond to the directions

$$w_1 = (\hat{x} + \hat{y} + \hat{z})/\sqrt{3}, \quad w_2 = (\hat{x} + \hat{y} - \hat{z})/\sqrt{3}, \quad w_3 = (-\hat{x} - \hat{y} - \hat{z})/\sqrt{3}.$$

That is, find $|w^+\rangle$ for each direction, with numerical values for the coefficients of $|0\rangle$ and $|1\rangle$ (three decimal places is sufficient). Check that $\langle w_3|w_1\rangle$ has the value you expect.

b) Find the directions, θ and ϕ, on the Bloch sphere corresponding to the following kets:

$$|\psi_1\rangle = -|0\rangle + 1/\sqrt{3}|1\rangle$$
$$|\psi_2\rangle = -i|0\rangle + 1/\sqrt{3}|1\rangle$$
$$|\psi_3\rangle = (1+i)|0\rangle + (1-i)|1\rangle$$

3. a) Let T be the time transformation operator from t_0 to t_1, and suppose that at t_0 a qubit is in the state $|0\rangle$. Find a unitary transformation T such that at t_1 the probabilities of being in the state $|0\rangle$ and $|1\rangle$ are

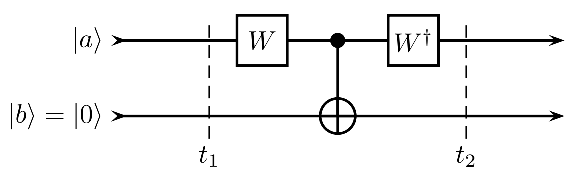
$$\Pr(0_1|0_0) = 2/3; \quad \Pr(1_1|0_0) = 1/3.$$

Here the subscripts label the time. (The answer is not unique, but a simple choice will simplify the next question.)

b) Describe T as a rotation on the Bloch sphere: some angle in a particular sense about some axis.

c) For the same unitary compute $\Pr(0_1|1_0)$ and $\Pr(1_1|1_0)$, i.e., the probabilities if the initial state is $|1\rangle$. If you have done the calculation correctly there will be a suprising connection between these numbers and those in (a). Show that this connection is not accidental, but is implied by the unitarity of T.

4. a) Show that the following circuit can be used to measure qubit a in the orthonormal basis $|w^+\rangle, |w^-\rangle$ provided W is chosen appropriately, assuming a pointer basis $|0\rangle, |1\rangle$ for qubit b. (You may find it convenient to express W in terms of dyads.)



Assume an initial state

$$|\Psi_0\rangle = \Big(\alpha|w^+\rangle + \beta|w^-\rangle\Big) \otimes |0\rangle,$$

at t_0. Indicate a consistent family of histories (and show that the family is consistent) so that the measurement outcome, $[0]_b$ or $[1]_b$, at time t_2 is appropriately correlated with properties of qubit a at time t_1 just before the measurement.

b) What are the states of a at t_2 conditioned on the b pointer states at this time?

c) Would your answers to (a) and (b) be different if the $W^\dagger$ gate were removed (set equal to I)?