ANNOUNCEMENT. There will be an hour exam on Tuesday afternoon, February 18, beginning at 3:00 pm (usual class hour). It is closed book, closed notes, no pocket calculators. Bring a pen or pencil. There will be no quantum information seminar that afternoon.

The exam will be based on the material in Assignments 1 to 4, and in the lectures up to and including Thursday, Feb. 6. However, the latter part of that lecture dealing with fully entangled states and dense coding is not included.

Problems 1 below is the first hour exam in a previous course. The others are practice problems. None are to be turned in. Solutions will be provided on Thursday, Feb. 13.

READING associated with recent and current lectures:
CLRS = Cormen, Leiserson, Rivest, and Stein, Introduction to Algorithms (Library reserve)
DPV = Dasgupta, Papadimitriou, and Vazirani, Algorithms (Library reserve)
QCQI = Nielsen and Chuang, Quantum Computation and Quantum Information
QIT = “Quantum Information Types” Course web site
TELEPORT = “Dense Coding, Teleportation, No Cloning” Course web site

Types of quantum information: QIT. There is also a published paper “Types of Quantum Information” available on the course web site, which is much longer and more detailed. The relevant sections are I, II, and III.

Dense Coding: QCQI Sec. 2.3; TELEPORT; The original paper by Bennett and Wiesner, reference in QCQI p. 119, is also worth reading
No cloning: QCQI Sec. 12.1, and Proposition 12.18 in Sec. 12.6.3; TELEPORT

READING AHEAD:
Cryptography: Stinson, Cryptography: theory and practice, Ch. 1; QCQI Sec. 12.6.1.
Number-theoretic algorithms: CLRS Ch. 31; DPV Ch. 1; QCQI App. 4
Complexity theory: CLRS Ch. 34; DPV Ch. 8; QCQI Ch. 3

EXERCISES (Not to be turned in)

1: From a previous hour exam.

INSTRUCTIONS. This examination consists of one question with 6 parts, each of which is worth between 15 and 20 points. If you cannot make progress on one part, take a look at a later part, for you may be able to do something with it. As on all examinations, it is important to give a (brief!) indication of your reasoning. Answer in a way which shows that you understand the subject, you really do know what you are doing.

There is a short list of formulas on the back side. If you think something else should be there, ask the instructor.

The circuit in the figure results in the state

$$|\Psi_2\rangle = \sqrt{1/3} |01\rangle + \sqrt{2/3} |10\rangle$$

at time $t_2$.

a) Find $|0_a\rangle \cdot |\Psi_2\rangle$ and $|1_a\rangle \cdot |\Psi_2\rangle$ (where as usual we think of $|0_a\rangle$ as denoting $|0\rangle_a \otimes I_b$) and use these to calculate the probabilities that qubit $a$ at time $t_2$ is in the state $|0\rangle$ or $|1\rangle$.

b) Find the reduced density operator $\rho_a$ for qubit $a$ at time $t_2$. What is its geometrical representation as a point inside the Bloch sphere? (i.e., where in the Bloch sphere is the point located?)
c) Use $\rho_\alpha$ to calculate the same probabilities you calculated in (a), indicating how you use $\rho_\alpha$ to obtain them.

d) Starting from $|\Psi_2\rangle$ use the circuit in the figure to calculate $|\Psi_1\rangle$ and $|\Psi_0\rangle$ at the earlier times $t_1$ and $t_0$.

e) Consider a family of histories for the times $t_0 < t_1 < t_2$ of the form

$$[\Psi_0] \otimes \{P, \tilde{P}\} \otimes \{Q, \tilde{Q}\},$$

where $\tilde{P} = I - P$ and $\tilde{Q} = I - Q$, thus four histories, all with the same initial state $|\Psi_0\rangle$. Explain why this family is consistent if $P = [\Psi_1]$, whatever the choice of the projector $Q$. You do not have to (and should not) make any reference to the specific $|\Psi_1\rangle$ you computed in (d).

f) Show that the family in (e) is consistent if $P = [+]_a$ and $Q = [0]_a$, $(|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2})$.

2. a) Find the normalized kets, expressed as linear combinations of $|0\rangle$ and $|1\rangle$, which in the Bloch sphere (spin half) picture correspond to the directions

$$w_1 = (\hat{x} + \hat{y} + \hat{z})/\sqrt{3}, \quad w_2 = (\hat{x} + \hat{y} - \hat{z})/\sqrt{3}, \quad w_3 = (-\hat{x} - \hat{y} - \hat{z})/\sqrt{3}.$$ 

That is, find $|w^\pm\rangle$ for each direction, with numerical values for the coefficients of $|0\rangle$ and $|1\rangle$ (three decimal places is sufficient). Check that $\langle w_3|w_1\rangle$ has the value you expect.

b) Find the directions, $\theta$ and $\phi$, on the Bloch sphere corresponding to the following kets:

$$|\psi_1\rangle = -|0\rangle + 1/\sqrt{3}|1\rangle$$

$$|\psi_2\rangle = -i|0\rangle + 1/\sqrt{3}|1\rangle$$

$$|\psi_3\rangle = (1 + i)|0\rangle + (1 - i)|1\rangle$$

3. a) Let $T$ be the time transformation operator from $t_0$ to $t_1$, and suppose that at $t_0$ a qubit is in the state $|0\rangle$. Find a unitary transformation $T$ such that at $t_1$ the probabilities of being in the state $|0\rangle$ and $|1\rangle$ are

$$\Pr(0_1|0_0) = 2/3; \quad \Pr(1_1|0_0) = 1/3.$$ 

Here the subscripts label the time. (The answer is not unique, but a simple choice will simplify the next question.)

b) Describe $T$ as a rotation on the Bloch sphere: some angle in a particular sense about some axis.

c) For the same unitary compute $\Pr(0_1|1_0)$ and $\Pr(1_1|1_0)$, i.e., the probabilities if the initial state is $|1\rangle$. If you have done the calculation correctly there will be a surprising connection between these numbers and those in (a). Show that this connection is not accidental, but is implied by the unitarity of $T$.

4. a) Show that the following circuit can be used to measure qubit $a$ in the orthonormal basis $|w^+\rangle, |w^-\rangle$ provided $W$ is chosen appropriately, assuming a pointer basis $|0\rangle, |1\rangle$ for qubit $b$. (You may find it convenient to express $W$ in terms of dyads.)

Assume an initial state

$$|\Psi_0\rangle = (\alpha|w^+\rangle + \beta|w^-\rangle) \otimes |0\rangle,$$

at $t_0$. Indicate a consistent family of histories (and show that the family is consistent) so that the measurement outcome, $|0\rangle_b$ or $|1\rangle_b$, at time $t_2$ is appropriately correlated with properties of qubit $a$ at time $t_1$ just before the measurement.

b) What are the states of $a$ at $t_2$ conditioned on the $b$ pointer states at this time?

c) Would your answers to (a) and (b) be different if the $W^\dagger$ gate were removed (set equal to $I$)?