

33-658, 758 Quantum Computation and Information Spring Semester, 2008
 Assignment No. 4. Due Feb. 11

DCTNC = Griffiths, Dense Coding, Teleportation, No Cloning (handed out in class)
 QCGM = Griffiths, Quantum Channels and Generalized Measurements (handed out in class)
 QCQI = Nielsen and Chuang, Quantum Computation and Quantum Information

READING:

Quantum Channels and Operations: QCQI Secs. 8.1, 8.2 except 8.2.4; 11.3.0, 11.3.2, Ch. 12, part preceding 12.1.1; QCGM Secs. 1-3.

POVMs: QCQI Sec. 2.2.6; QCGM Sec. 4

Dense coding: QCQI Sec. 2.3; DCTNC Secs. 1,2

READING AHEAD:

Teleportation: QCQI Secs. 1.3.7, 4.4. (The original paper by Bennett et al., reference in QCQI p. 59, is also worth reading.) DCTNC Sec. 3

No Cloning: QCQI Sec. 12.1, and Proposition 12.18 in Sec. 12.6.3; DCTNC Sec. 4

Deutsch-Jozsa algorithm: QCQI Secs. 1.4.3, 1.4.4

Computational complexity: QCQI Ch. 3; Cormen, Leiserson, Rivest, *Introduction to Algorithms*, 1st edition, Ch. 33, Number-Theoretic Algorithms, and Ch. 34, NP-Completeness. These are Chs. 31 and 34 in the 2d edition of the book, by Cormen, Leiserson, Rivest and Stein.

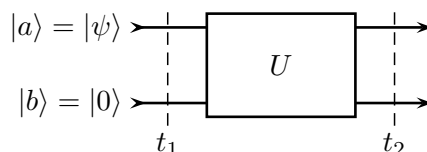
EXERCISES:

1. Report

Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course. You will find a sample at the end of the problem set.

2. Kraus and POVM operators for specified 2 qubit unitary; also find circuit

A model measurement is represented by the circuit:



The unitary time development matrix is

$$U = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \end{pmatrix}.$$

a) Find the Kraus operators K_0 and K_1 such that

$$U(|\psi\rangle \otimes |0\rangle) = \sum_m (K_m|\psi\rangle) \otimes |m\rangle,$$

and write them as 2×2 matrices.

b) Find the corresponding POVM operators $Q_m = K_m^\dagger K_m$ and show that they correspond to an ideal measurement of some quantity, which you should identify. Check that when the input $|\psi\rangle$ is an eigenstate of this quantity you get the expected pointer result with probability 1.

c) What can you say about the state of qubit a at time t_2 conditioned on the outcomes $[0b]$ and $[1b]$ at this time?

d) Find a quantum circuit using single qubit gates and some CNOTs that will produce the unitary transformation U . [Hint. The answer is not unique. Thinking about what the measurement measures may be helpful. There is a circuit containing two CNOTs, along with an H and an X , that does the trick.]

3. DO NOT HAND IN. Properties of superoperator expressed using Kraus operators

Let the superoperator \mathcal{E} be defined by

$$\mathcal{E}(A) = \sum_j E_j A E_j^\dagger, \quad \sum_j E_j^\dagger E_j = I.$$

Show directly from the definition that it has the following properties:

- i) $\mathcal{E}(A^\dagger) = [\mathcal{E}(A)]^\dagger$.
- ii) If A is a positive operator, $\mathcal{E}(A)$ is a positive operator.
- iii) $\text{Tr}[\mathcal{E}(A)] = \text{Tr}(A)$.

4. Bloch sphere mapping produced by two Kraus operators

Show that the operators

$$E_0 = \sqrt{p} I, \quad E_1 = \sqrt{1-p} \sigma_y,$$

with $0 \leq p \leq 1$, satisfy the condition $\sum_j E_j^\dagger E_j = I$, and thus define a completely-positive trace-preserving superoperator \mathcal{E} . Find the matrix S_{kj} such that $\mathcal{E}(\sigma_j) = \sum_k S_{kj} \sigma_k$, and use it to construct the explicit mapping of the Bloch sphere $\mathbf{r} \rightarrow \mathbf{r}'$ produced by \mathcal{E} . Then describe this mapping in words: shrinkage in what direction; what sort of rotation? For what value of p is the shrinkage the largest (the resulting ellipsoid the smallest)? What points inside or on the surface of the Bloch sphere are left invariant (mapped to themselves)?

5. DO NOT HAND IN. Kraus operators and superoperator for channel based on controlled-not

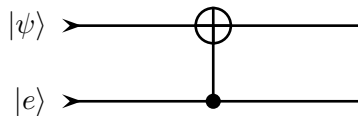
Consider the channel for the a qubit, initially in a state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

produced by the following circuit, where the environment e qubit is initially in the state

$$|e\rangle = \sqrt{1-p}|0\rangle + \sqrt{p}|1\rangle,$$

with p between 0 and 1.



a) Work out the Kraus operators (mapping \mathcal{H}_a to \mathcal{H}_a) assuming a standard $\{|0\rangle, |1\rangle\}$ basis for the e qubit after the interaction, and check that $\sum_l K_l^\dagger K_l = I$. Write the Kraus operators both as 2×2 matrices in the standard basis, and as sums of Pauli matrices.

b) Now find the Kraus operators assuming a basis $\{|+\rangle, |-\rangle\}$ for the e qubit after the interaction, and show that these are linear combinations of the ones you found in (a) with coefficients that form a unitary matrix.

c) Find the channel superoperator \mathcal{S} by finding the matrix S_{jk} such that in terms of Pauli matrices, with $\sigma_0 = I$,

$$\mathcal{S}(\sigma_k) = \sum_j S_{jk} \sigma_j.$$

Check that both sets of Kraus operators, from (a) and (b), give rise to the same superoperator.

d) For $p = 0$, $p = 1/2$, and $p = 1$, describe what the channel does in terms of how it maps the original Bloch sphere by shrinkage along different directions and (possible) rotation.

6. Three outcome POVM for a qubit

Alice prepares a qubit randomly in one of the three states $|\psi_j\rangle$:

$$|\psi_1\rangle = (|0\rangle + |1\rangle)/\sqrt{2}, \quad |\psi_2\rangle = (|0\rangle + e^{i2\pi/3}|1\rangle)/\sqrt{2}, \quad |\psi_3\rangle = (|0\rangle + e^{-i2\pi/3}|1\rangle)/\sqrt{2},$$

and sends it to Carol, who on the basis of whatever measurements she wants to carry out is to specify a number $k = 1$ or 2 or 3 in such a way that k is *not* equal to j . Carol wins \$1 every time k is unequal to j , but loses \$10 every time $k = j$.

a) How much, on average, will Carol win or lose if she measures in the S_x basis $(|0\rangle \pm |1\rangle)/\sqrt{2}$ every time and makes optimal use of this information?

b) Find a collection $\{|\omega_j\rangle\}$, $j = 1, 2, 3$, such that $\langle\omega_j|\psi_j\rangle = 0$, and so that with $G_j = |\omega_j\rangle\langle\omega_j|$ up to a constant of proportionality, $\sum_j G_j = I$, and thus the $\{G_j\}$ form a POVM. Write down each G_j as a 2×2 matrix in the standard representation, and check that the matrices add up to I .

c) If Carol uses the $\{G_j\}$ POVM and makes optimal use of this information, how much on average will she win or lose?

7. Dense coding for $d = 3$, to send 9 messages

Dense coding is possible using a tensor product $\mathcal{H} = \mathcal{A} \otimes \mathcal{B}$ of two spaces of equal dimension d for $d > 2$. What is needed is a basis of \mathcal{H} consisting of fully-entangled states with the property that it is possible to map any basis state to any other basis state using unitary operations on \mathcal{A} alone, or on \mathcal{B} alone. Consider the case $d = 3$.

a) Show that the nine states

$$|\psi_{jk}\rangle = (|j, 0\rangle + \omega^k |j + 1, 1\rangle + \omega^{2k} |j + 2, 2\rangle)/\sqrt{3}$$

for $0 \leq j, k \leq 2$ are orthonormal and maximally entangled, and thus form a basis. Here the standard basis kets of \mathcal{A} and \mathcal{B} are denoted by $|0\rangle, |1\rangle, |2\rangle$; $|01\rangle$ or $|0, 1\rangle$ denotes $|0\rangle \otimes |1\rangle$; $\omega = e^{2\pi i/3}$ is a cube root of 1; and the quantities $j + 1$, $j + 2$, and $2k$ on the right side of the definition are understood as mod 3. E.g., when $k = 2$, $2k$ is to be interpreted as $4 - 3 = 1$. [Hint: What is $1 + \omega + \omega^2$?]

b) Show that Alice can map $|\psi_{00}\rangle$ onto any basis state $|\psi_{jk}\rangle$, up to a phase factor $e^{i\phi}$ which could depend on j and k , by means of suitable unitaries acting on \mathcal{A} alone. In order to be systematic, let these unitaries be produced by two variable gates, one after the other, each gate can represent 3 different unitaries on \mathcal{A} .

c) Describe briefly how dense coding will function if Alice and Bob initially share the fully entangled $|\psi_{00}\rangle$, and after applying the pair of unitaries in (b) Alice sends \mathcal{A} to Bob over a $d = 3$ quantum channel. Indicate what measurements Bob has to do, but save the details for (d) below.

d) Construct a “disentangling” circuit acting on $\mathcal{A} \otimes \mathcal{B}$ which can be used for the measurement step in (c) in the sense that after this circuit acts, Alice’s message can be read out by measurements carried out separately on \mathcal{A} and \mathcal{B} in the standard basis. (That is, this circuit corresponds to the final steps occurring in Bob’s laboratory in the case of qubit dense coding, as discussed in Sec. 10.2 of the notes.) As in the qubit case, use one controlled-unitary gate with either \mathcal{A} or \mathcal{B} acting as the control, and the other as the data, and another unitary gate which acts only on \mathcal{A} , or only on \mathcal{B} . These are gates controlled by or acting on $d = 3$ Hilbert spaces, for which there is no standard terminology; so describe them in your own words. [Hint: One way to proceed is to construct a circuit that maps $|j\rangle \otimes |k\rangle$ to $|\psi_{jk}\rangle$. Its inverse is the desired disentangling circuit.]