33-658, 758 Quantum Computation and Information Spring Semester, 2014
Assignment No. 4. Due Tuesday, February 11

ANNOUNCEMENT. There will be an hour exam on Tuesday afternoon, February 18, beginning at 3:00 pm (usual class hour). It is closed book, closed notes, no pocket calculators. Bring a pen or pencil. There will be no quantum information seminar that afternoon.

READING:
CQT = Griffiths, Consistent Quantum Theory
DENSE = "Density Operators and Ensembles" Course web site
TELEPORT = "Dense Coding, Teleportation, No Cloning" Course web site
INFORMATION = "Classical Information Theory" Course web site
QCQI = Nielsen and Chuang, Quantum Computation and Quantum Information
QIT = "Quantum Information Types" Course web site

Correlations: CQT Ch. 15; DENSE Sec. 2; QCQI Sec. 2.4

Information theory: INFORMATION; QCQI Secs. 11.1, 11.2; Cover and Thomas has relevant material, but at a rather advanced level, in Chs. 2 and 7.

Types of quantum information: QIT; The published paper TQI (also on the course web site) is much longer and more detailed; the relevant sections are I, II, and III.

READING AHEAD:
No cloning: QCQI Sec. 12.1, and Proposition 12.18 in Sec. 12.6.3; TELEPORT

Dense Coding: QCQI Sec. 2.3; TELEPORT; The original paper by Bennett and Wiesner, reference in QCQI p. 119, is also worth reading

Teleportation: QCQI Secs. 1.3.7, 4.4; TELEPORT; Bennett et al., Phys. Rev. Lett. 70 (1993) 1895 (Original paper). If you are ambitious, take a look at R. B. Griffiths, Phys. Rev. A 66 (2002) 012311, also in http://arxiv.org/archive/quant-ph/0203058.

Cryptography: Stinson, *Cryptography: theory and practice*, Ch. 1; QCQI Sec. 12.6.1.

EXERCISES:

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course.

1b. Only for students enrolled in 33-758 (Pitt 4 credits): Summarize in half a page to a page what you learned from the most recent seminar.

2. a) Suppose a qubit described by a density operator

$$\rho_0 = \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} /2$$

at t_0 undergoes a unitary time development corresponding to

$$T(t_1, t_0) = \begin{pmatrix} i\sqrt{3}/2 & -1/2 \\ -1/2 & i\sqrt{3}/2 \end{pmatrix}.$$

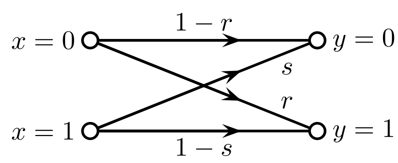
Find ρ_1 at t_1 as a 2×2 matrix, and also find $\bar{x}$, $\bar{y}$, and $\bar{z}$ as a function of x, y, and z if ρ_1 is written in the form

$$\rho_1 = \begin{pmatrix} 1+\bar{z} & \bar{x}-i\bar{y} \\ \bar{x}+i\bar{y} & 1-\bar{z} \end{pmatrix} /2.$$

b) From the dependence of $(\bar{x}, \bar{y}, \bar{z})$ on (x, y, z) deduce that the unitary transformation $T(t_1, t_0)$ corresponds to the rotation of the Bloch sphere by a particular angle about a particular axis. [Hint. Consider what happens to $(1, 0, 0)$ and $(0, 1, 0)$.]

c) You will find a formula on p. 174 of QCQI for the unitary corresponding to this rotation. Is it the same as $T(t_1, t_0)$ given above? Explain any difference.

3. Consider a classical channel with the conditional probabilities shown in the figure (see Nielsen and Chuang, p. 354).



That is, $p(y \,|\, x)$ is given by

$$p(0\,|\,0) = 1 - r, \quad p(1\,|\,0) = r, \quad p(0\,|\,1) = s, \quad p(1\,|\,1) = 1 - s,$$

where r and s are between 0 and 1. Assume that $p(x = 0)$ and $p(x = 1)$ are both equal to 1/2.

a) Find the joint probability distribution $p(x, y)$ the marginal distribution $p(y)$, and the conditional distribution $p(x \,|\, y)$ as functions of the parameters r and s, and check that $\sum_y p(y)$ and $\sum_x p(x \,|\, y)$ are equal to 1.

b) Find expressions for $H(X)$, $H(Y)$, $H(X, Y)$, $H(X|Y)$, $H(Y|X)$ as functions of r and s, and show that

$$\begin{aligned} I(X{:}Y) = \log 2 + \tfrac{1}{2}\{ & r \log r + (1 - r)\log(1 - r) + s \log s + (1 - s)\log(1 - s) \\ & - (1 - r + s)\log(1 - r + s) - (1 + r - s)\log(1 + r - s)\}. \end{aligned}$$

c) For each of the following particular cases evaluate $H(Y)$, $H(X, Y)$, $H(X|Y)$, $H(Y|X)$ and $I(X{:}Y)$, and whenever one of these quantities is zero, provide a brief explanation of why this seems reasonable, in terms of missing information or information transmitted, etc.

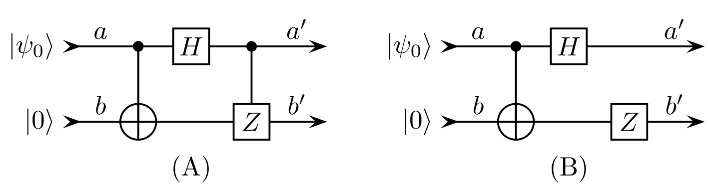
i) $r = s = 0$.
ii) $r = s = 1$.
iii) $r = 0$, $s = 1$.

d) This part is OPTIONAL. Do not turn in. For the case $r = 0$ and $s = 1/2$, evaluate $H(X \,|\, Y)$ by taking the difference between $H(X, Y)$ and $H(Y)$, and also by calculating $H(X|y)$ for each y and using the formula

$$H(X|Y) = \sum_y p(y) H(X|y).$$

If one of the $H(X|y) = 0$, does this seem reasonable in terms of what you would expect for missing information?

e) Find the necessary and sufficient conditions on r and s such that $I(X{:}Y) = 0$.



4. a) Show that the two qubit circuit shown in part (A) of the figure provides an ideal channel from a to b' (a perfect channel up to a possible one-qubit unitary) by working out what the circuit does if $|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$.

b) Trace the progress of Z information, which is to say the difference between $|0\rangle$ and $|1\rangle$ at the input a (i.e., the difference between the $\alpha = 1, \beta = 0$ and the $\alpha = 0, \beta = 1$ state) through the (A) circuit, indicating at each intermediate time in which of the qubits this information is located and what form it takes (i.e., it may be changed into some other type of information). Is it available at a'? Then do the same for X and Y types of information entering the circuit at a. Your discussion will be aided by computing partial traces of the quantum state down to each qubit at each intermediate time.

c) Now carry out a similar analysis for the circuit in part (B) of the figure, starting by working out the unitary time development of $|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$.

d) Relate the results obtained in (b) and (c) to the Exclusion theorem, letting a in the theorem be the initial state of qubit a in part (A) or (B) of the figure, and b and c the final states a' and b'.

Exclusion theorem: Let a, b, and c be three systems and let $\mathcal{V} = \{|v_j\rangle\}$, $\mathcal{W} = \{|w_k\rangle\}$ be two mutually-unbiased orthonormal bases of $\mathcal{H}_a$, i.e., for all j and k $|\langle v_j|w_k\rangle| = 1/\sqrt{d_a}$, where d_a is the dimension of $\mathcal{H}_a$. Then if the $\mathcal{V}$ information about a is perfectly present in b, the $\mathcal{W}$ information about a is completely absent from c.

5. The density operator for a qubit is given in the standard basis by the following matrix:

$$\rho = \begin{pmatrix} 1/2 & i/3 \\ -i/3 & 1/2 \end{pmatrix}.$$

Calculate the Shannon entropy $H(p)$ for a measurement in the standard basis, and compare this with the von Neumann entropy $S(\rho)$; give numerical values in bits (logarithm to base 2). Then show that the Shannon entropy for a measurement in an appropriate basis, which you should specify, is equal to the von Neumann entropy.