33-658, 758 Quantum Computation and Information Spring Semester, 2014
Assignment No. 3. Due Tuesday, February 4

ANNOUNCEMENT. There will be an hour exam on Tuesday afternoon, February 18, beginning at 3:00 pm (usual class hour). It is closed book, closed notes, no pocket calculators. Bring a pen or pencil. There will be no quantum information seminar that afternoon.

READING:
CQT = Griffiths, Consistent Quantum Theory
DENSE = "Density Operators and Ensembles" Course web site
HISTORIES = "Histories and Consistency" Course web site
INFORMATION = "Classical Information Theory" Course web site
MEASUREMENTS = "Measurements" Course web site
QCQI = Nielsen and Chuang, Quantum Computation and Quantum Information
QIT = "Quantum Information Types" Course web site
TQI = "Types of Quantum Information" Course web site

Quantum histories: CQT Ch. 8; HISTORIES

Consistency conditions: CQT Sec. 11.5; HISTORIES. (The general treatment given in CQT Ch. 10 is not needed for this course.)

Measurements: CQT Secs. 7.4; 17.1, 17.2, 18.1, 18.2, 18.3; MEASUREMENTS
The axiomatic approach in QCQI Secs. 2.2.3 and the following subsections is correct as a method of calculation, but not too easy to understand.

Density operators and ensembles: CQT Ch. 15; DENSE; QCQI Sec. 2.4

READING AHEAD:
Information theory: INFORMATION; QCQI Secs. 11.1, 11.2; Cover and Thomas has relevant material, but at a rather advanced level, in Chs. 2 and 7.
Types of quantum information: QIT; TQI

EXERCISES:

1. Turn in at most one page, and not less than half a page, indicating what you have read, examples or exercises (apart from those assigned below) that you worked out, difficulties you encountered, questions that came to mind, etc. You may include complaints about the course.

1b. Only for students enrolled in 33-758: Summarize in half a page to a page what you learned from the most recent seminar.

2. a) Show that the family of histories with the structure

$$[+] \odot \{[0],[1]\} \odot \{[+],[-]\}$$

and trivial dynamics, $T(t',t) = I$, is inconsistent. That is, the initial state at t_0 is $|\psi_0\rangle = |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, and the decomposition of the identity is $\{[0],[1]\}$ at t_1 and $\{[+],[-]\}$ at t_2.

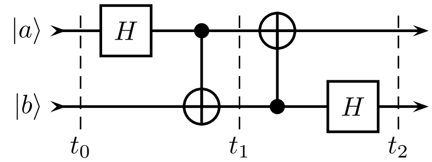
b) Show that the same family of histories may be made consistent by introducing a nontrivial $T(t_1,t_0)$ in place of I, while keeping $T(t_2,t_1) = I$, or by keeping $T(t_1,t_0) = I$ and introducing a nontrivial $T(t_2,t_1)$. Find appropriate nontrivial unitaries for the two cases (the answer is not unique).

c) Suppose that

$$T(t_2,t_1) = T(t_1,t_0) = \begin{pmatrix} \sqrt{1/3} & \sqrt{2/3} \\ -\sqrt{2/3} & \sqrt{1/3} \end{pmatrix},$$

and the initial $|\psi_0\rangle = |+\rangle$, as in (a). Find decompositions of the identity at t_1 and t_2 such that the resulting family will be consistent. There are several possibilities, and you should identity at least two (i.e., two pairs of decompositions; you might use the same decomposition at t_1 and two different ones at t_2).

3. Consider the circuit



and assume the initial state is $|\psi_0\rangle = |00\rangle$.

a) Work out the unitary time development of $|\psi_0\rangle$ and compute the probabilities of the four histories

$$[\psi_0] \odot I \odot \{[jk]\},$$

where the states $\{|jk\rangle\}$ at t_2, j and k equal to 0 or 1, are elements of the standard basis. (The family is consistent, since only two times are involved.)

b) Show that if one replaces I at t_1 with the decomposition $\{[0a], [1a]\}$, i.e., qubit a is either $|0\rangle$ or $|1\rangle$, the resulting family is inconsistent, and similarly if I is replaced with $\{[0b], [1b]\}$.

c) Optional. Extend (b) by proving that using *any* decomposition of the identity corresponding to an orthonormal basis of qubit a at t_1 results in an inconsistent family, and the same for qubit b.

d) Show that the family with $\{[0a], [1a]\}$ at t_1 can be made consistent by replacing $\{|jk\rangle\}$ at t_2 with a different orthonormal basis for the two qubit system, one which is still of the product type $\{|a_j\rangle|b_k\rangle\}$.

4. a) Which of the following matrices represent density operators? Give reasons for your answers.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}, \quad \begin{pmatrix} 3/4 & 1/2 \\ 1/2 & 1/4 \end{pmatrix}, \quad \begin{pmatrix} 3/4 & -i/4 \\ i/4 & 1/4 \end{pmatrix}, \quad \begin{pmatrix} 1/2 & (1+i)/4 \\ (1+i)/4 & 1/2 \end{pmatrix}$$

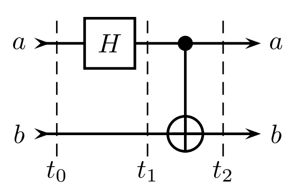
b) In those cases in which the matrix corresponds to a density operator, find the corresponding Bloch sphere vector $\boldsymbol{r} = (x, y, z)$.

5. a) Suppose that that all you know about some qubit is that it was prepared in the state $|0\rangle$ with probability 1/3, in the state $|1\rangle$ with probability 1/3, and in the state $|y^+\rangle$ with probability 1/3. What density operator should you assign to it? Express your answer as a matrix in the standard basis, and also find the corresponding vector $\vec{r} = (x, y, z)$ in the Bloch sphere representation; see QCQI, p. 105, (2.175).

b) Suppose that a measurement of σ_y is carried out on this qubit. Obtain the probabilities for $\sigma_y = \pm 1$ by writing the density operator as a matrix in the $|y^+\rangle$, $|y^-\rangle$ basis. Then calculate the average $\langle \sigma_y \rangle$ by evaluating $\mathrm{Tr}(\rho\sigma_y)$ in the standard basis. Are these results consistent? (If not, you have made a mistake.)

c) Assuming a density operator ρ, find an expression for $\langle \sigma_y \rangle = \mathrm{Tr}(\rho\sigma_y)$ in terms of the Bloch sphere vector $\vec{r}$ associated with ρ.

6. Consider the two-qubit circuit



a) Given an initial state

$$|\psi_0\rangle = \big(\alpha|0\rangle + \beta|1\rangle\big) \otimes |0\rangle$$

for the two qubits a and b at t_0, find $|\psi_1\rangle$ and $|\psi_2\rangle$ at t_1 and t_2, and use these together with the Born rule to calculate the probabilities $\Pr([0]_a)$, $\Pr([1]_a)$, $\Pr([0]_b)$, $\Pr([1]_b)$ at t_1 and at t_2, where $[0]_a$ is $[0] = |0\rangle\langle 0|$ for qubit a, etc.

b) Write down the dyads $|\psi_1\rangle\langle\psi_1|$ and $|\psi_2\rangle\langle\psi_2|$, either as a sum of dyads using states in the standard basis or as a matrix in the standard basis, and from these compute the reduced density operators ρ_a and ρ_b for qubits a and b, again as sums of dyads or as matrices in the standard basis, at the times t_1 and t_2. Use these reduced density operators to calculate the same probabilities as in (a).