Consistent Quantum Theory
Exercises for Ch. 4
(Version of 1 May 2003)

4.1 a) For a classical harmonic oscillator with energy

$$E(x,p) = p^2/2m + \tfrac{1}{2}Kx^2.$$

sketch regions in the x,p plane corresponding to the properties

$$P: \quad p_0 \le p \le p_1 \text{ for some } p_0 < 0 < p_1.$$
$$Q: \quad E \le E_0 \text{ for some } E_0 > 0.$$

b) Indicate the regions (on your previous sketch or on a new sketch) $P \wedge Q$ and $\tilde{P} \vee Q$, where $\tilde{P}$ is the negation of P.

c) Under what conditions (what choice of parameters p_0, p_1, E_0) will the property $\tilde{P} \wedge Q$ be always false, i.e., impossible?

4.2 a) For projectors X, (4.20) and X', (4.46), describe the projector $X \vee X'$, allowing various possibilities for the order of the four numbers x_1, x_2, x'_1, x'_2, assuming always that $x_1 < x_2$ and $x'_1 < x'_2$.

b) For the particular order $x_1 < x'_1 < x_2 < x'_2$ in (4.47), check the two relationships in (4.51) by separately working out both sides with $P = X$ and $Q = X'$, and showing that the projectors are equal.

4.3 Consider a toy model in one dimension, with $|m\rangle$ the ket for a particle at site m.

a) Write down as dyads of the form $|m\rangle\langle m'|$, or sums of the dyads, the following projectors:
P: Particle at site $m = 1$.
Q: Particle between 0 and 2. (Q projects onto a three-dimensional subspace).
R: Projector onto the ray (one-dimensional subspace) containing $|\phi\rangle = |1\rangle + 2i|3\rangle$.

b) Which of these projectors commute with each other and which do not commute? You may either work out the commutator, or give reasons why it is or is not zero.

c) In all cases in which two projectors commute, find the projectors corresponding to the conjunction $(A \wedge B)$ and to the disjunction $(A \vee B)$ of the two properties.

4.4 Let $|\phi_n\rangle$ be the state of a harmonic oscillator with energy $E = (n + \frac{1}{2})\hbar\omega$, and let

$$P = [\phi_0] + [\phi_1], \quad Q = [\phi_0] + [\phi_1] + [\phi_2]$$

be the projectors for the properties $E < 2\hbar\omega$ and $E < 3\hbar\omega$, respectively.

a) Find the projectors PQ, $\tilde{P}Q$, and $P\tilde{Q}$, and in each case explain *briefly* (one or two sentences) why the property corresponding to the product is what you would expect for the conjunction of the two properties. (E.g., "If the energy is less than $2\hbar\omega$ and also less than $3\hbar\omega$, then it is obvious that....")

b) Find a nonzero projector R *other than* $|0\rangle\langle 0|$, $|1\rangle\langle 1|$, or P such that $PR = R$. [Hint: Does every state in the subspace onto which P projects have a well-defined energy?]

c) Find a projector S such that $QS = S$, but $PS \neq SP$.

4.5 Show that for a spin half particle $[z^+]$ and $[x^+]$ do not commute, and then give an argument why the same will be true for the projectors $[v^+]$ and $[w^+]$ for the spin to be along any two directions v and w (unit vectors on the sphere), apart from certain exceptional cases, which you should specify.