Chapter 26

Decoherence and the Classical Limit

26.1 Introduction

Classical mechanics deals with objects which have a precise location and move in a deterministic way as a function of time. By contrast, quantum mechanics uses wave functions which always have some finite spatial extent, and the time development of a quantum system is (usually) random or stochastic. Nonetheless, most physicists regard classical mechanics as an approximation to quantum mechanics, an approximation which works well when the object of interest contains a large number of atoms. How can it be that classical mechanics emerges as a good approximation to quantum mechanics in the case of large objects?

Part of the answer to the question lies in the process of *decoherence* in which a quantum object or system interacting with a suitable environment (which is also quantum mechanical) loses certain types of quantum coherence which would be present in a completely isolated system. Even in classical physics the interaction of a system with its environment can have significant effects. It can lead to irreversible processes in which mechanical energy is turned into heat, with a resulting increase of the total entropy. Think of a ball rolling along a smooth, flat surface. Eventually it comes to rest as its kinetic energy is changed into heat in the surrounding air due to viscous effects, or dissipated as vibrational energy inside the ball or in the material which makes up the surface. (From this perspective the vibrational modes of the ball form part of its "environment".) While decoherence is (by definition) quantum mechanical, and so lacks any exact analog in classical physics, it is closely related to irreversible effects.

In this chapter we explore a very simple case, one might even think of it as a toy model, of a quantum particle interacting with its environment as it passes through an interferometer, in order to illustrate some of the principles which govern decoherence. In the final section there are some remarks on how classical mechanics emerges as a limiting case of quantum mechanics, and the role which decoherence plays in relating classical and quantum physics. The discussion of decoherence and of the classical limit of quantum mechanics presented here is only intended as an introduction to a complex subject. The bibliography indicates some sources of additional material.

26.2 Particle in an Interferometer

Consider a particle passing through an interferometer, shown schematically in Fig. 26.1, in which an input beam in channel a is separated by a beam splitter into two arms c and d, and then passes through a second beam splitter into two output channels e and f. While this has been drawn as a Mach-Zehnder interferometer similar to the interferometers considered in earlier chapters, it is best to think of it as a neutron interferometer or an interferometer for atoms. The principles of interference for photons and material particles are the same, but photons tend to interact with their environment in a different way.

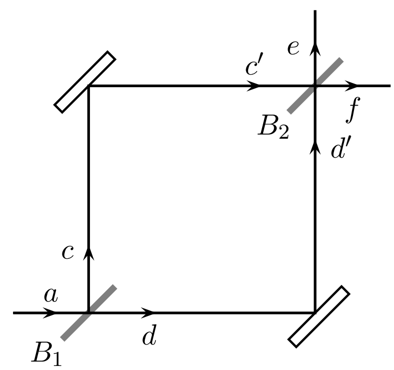

Figure 26.1: Particle passing through an interferometer.

Let us suppose that the interferometer is set up so that a particle entering through channel a always emerges in the f channel due to interference between the waves in the two arms c and d. As discussed in Ch. 13, this interference disappears if there is a measurement device in one or both of the arms which determines which arm the particle passes through. Even in the absence of a measuring device, the particle may interact with something, say a gas molecule, while traveling through one arm but not through the other arm. In this way the interference effect will be reduced if not entirely removed. One refers to this process as *decoherence* since it removes, or at least reduces the interference effects resulting from a *coherent superposition* of the two wave packets in the two arms. Sometimes one speaks metaphorically of the environment "measuring" which arm the particle passes through.

Assume that at the first beam splitter the particle state undergoes a unitary time development

$$|a\rangle \mapsto (|c\rangle + |d\rangle)/\sqrt{2}, \tag{26.1}$$

while passage through the arms of the interferometer results in

$$|c\rangle \mapsto |c'\rangle, \quad |d\rangle \mapsto |d'\rangle. \tag{26.2}$$

Here $|a\rangle$ is a wave packet in the input channel at time t_0, $|c\rangle$ and $|d\rangle$ are wave packets emerging from the first beam splitter in the c and d arms of the interferometer at time t_1, and $|c'\rangle$ and $|d'\rangle$ are the corresponding wave packets at time t_2 just before they reach the second beam splitter. The effect of passing through the second beam splitter is represented by

$$|c'\rangle \mapsto \big(|e\rangle + |f\rangle\big)/\sqrt{2}, \quad |d'\rangle \mapsto \big(-|e\rangle + |f\rangle\big)/\sqrt{2}, \tag{26.3}$$

where $|e\rangle$ and $|f\rangle$ are wave packets in the output channels of the second beam splitter at time t_3. The notation is chosen to resemble that used for the toy models in Sec. 12.1 and Ch. 13.

Next assume that while inside the interferometer the particle interacts with something in the environment in a way which results in a unitary transformation of the form

$$|c\rangle|\epsilon\rangle \mapsto |c'\rangle|\epsilon'\rangle, \quad |d\rangle|\epsilon\rangle \mapsto |d'\rangle|\epsilon''\rangle, \tag{26.4}$$

on the Hilbert space $\mathcal{A} \otimes \mathcal{E}$ of the particle $\mathcal{A}$ and environment $\mathcal{E}$, where $|\epsilon\rangle$ is the normalized state of $\mathcal{E}$ at time t_1, and $|\epsilon'\rangle$ and $|\epsilon''\rangle$ are normalized states at t_2. For example, it might be the case that if the particle passes through the c arm some molecule is scattered from it resulting in the change from $|\epsilon\rangle$ to $|\epsilon'\rangle$, whereas if the particle passes through the d arm there is no scattering, and the change in the environment from $|\epsilon\rangle$ to $|\epsilon''\rangle$ is the same as it would have been in the absence of the particle. The complex number

$$\alpha = \langle\epsilon''|\epsilon'\rangle = \alpha' + i\alpha'', \tag{26.5}$$

with real and imaginary parts α' and α'', plays an important role in the following discussion. The final particle wave packets $|c'\rangle$ and $|d'\rangle$ in (26.4) are the same as in the absence of any interaction with the environment, (26.2). That is, we are assuming that the scattering process has an insignificant influence upon the center of mass of the particle itself as it travels through either arm of the interferometer. This approximation is made in order to simplify the following discussion; one could, of course, explore a more complicated situation.

The complete unitary time evolution of the particle and its environment as the particle passes through the interferometer is given by

$$\begin{aligned} |\psi_0\rangle &= |a\rangle|\epsilon\rangle \mapsto \big(|c\rangle + |d\rangle\big)|\epsilon\rangle/\sqrt{2} \mapsto \\ |\psi_2\rangle &= \big(|c'\rangle|\epsilon'\rangle + |d'\rangle|\epsilon''\rangle\big)/\sqrt{2} \mapsto \\ |\psi_3\rangle &= \big\{|e\rangle\big(|\epsilon'\rangle - |\epsilon''\rangle\big) + |f\rangle\big(|\epsilon'\rangle + |\epsilon''\rangle\big)\big\}/2, \end{aligned} \tag{26.6}$$

where we assume that the environment state $|\epsilon\rangle$ does not change between t_0 and t_1. (This is not essential, and one could assume a different state, say $|\bar{\epsilon}\rangle$, at t_0, which develops unitarily into $|\epsilon\rangle$ at t_1.) Therefore, in the family with support $[\psi_0] \odot I \odot I \odot \{[e], [f]\}$ the probabilities for the particle emerging in each of the output channels are given by

$$\begin{aligned} \Pr(e) &= \tfrac{1}{4}\big(\langle\epsilon'| - \langle\epsilon''|\big) \cdot \big(|\epsilon'\rangle - |\epsilon''\rangle\big) \\ &= \tfrac{1}{4}\big(\langle\epsilon'|\epsilon'\rangle + \langle\epsilon''|\epsilon''\rangle - \langle\epsilon'|\epsilon''\rangle - \langle\epsilon''|\epsilon'\rangle\big) = \tfrac{1}{2}(1 - \alpha'), \\ \Pr(f) &= \tfrac{1}{4}\big(\langle\epsilon'| + \langle\epsilon''|\big) \cdot \big(|\epsilon'\rangle + |\epsilon''\rangle\big) = \tfrac{1}{2}(1 + \alpha'). \end{aligned} \tag{26.7}$$

Because the states entering the inner product in (26.5) are normalized, $|\alpha|$ cannot be greater than 1. If $|\epsilon''\rangle = |\epsilon'\rangle$, then $\alpha' = \alpha = 1$ and there is no decoherence: the interference pattern is the same as in the absence of any interaction with the environment, and the particle always emerges in f. The interference effect disappears when $\alpha' = 0$, and the particle emerges with equal probability in e or f. This could happen even with $|\alpha|$ rather large, for example, $\alpha = i$. But in such a case there would still be a substantial coherence between the wave packets in the c and d arms, and the corresponding interference effect could be detected by shifting the second beam splitter by a small amount so as to change the difference in path length between the c and d arms by a quarter wavelength. Hence it seems sensible to use $|\alpha|$ rather than α' as a measure of coherence between the two arms of the interferometer, and $1 - |\alpha|$ as a measure of the amount of decoherence.

26.3 Density Matrix

In a situation in which one is interested in what happens to the particle after it passes through the second beam splitter without reference to the final state of the environment, it is convenient to use a density matrix ρ_2 for the particle at the intermediate time t_2 in (26.6), just before the particle passes through the second beam splitter. By taking a partial trace over the environment $\mathcal{E}$ in the manner indicated in Sec. 15.3, one obtains

$$\rho_2 = \mathrm{Tr}_{\mathcal{E}}\bigl(|\psi_2\rangle\langle\psi_2|\bigr) = \tfrac{1}{2}\bigl(|c'\rangle\langle c'| + |d'\rangle\langle d'| + \alpha|c'\rangle\langle d'| + \alpha^*|d'\rangle\langle c'|\bigr). \tag{26.8}$$

This has the form

$$\rho_2 = \begin{pmatrix} 1/2 & \alpha/2 \\ \alpha^*/2 & 1/2 \end{pmatrix} \tag{26.9}$$

when written as a matrix in the basis $\{|c'\rangle, |d'\rangle\}$, with $\langle c'|\rho_2|c'\rangle$ in the upper left corner. If we think of ρ_2 as a pre-probability, see Sec. 15.2, its diagonal elements represent the probability that the particle will be in the c or the d arm. Twice the magnitude of the off-diagonal elements serves as a convenient measure of coherence between the two arms of the interferometer, and thus $1 - |\alpha|$ is a measure of the decoherence.

After the particle passes through the second beam splitter, the density matrix is given by (see Sec. 15.4)

$$\rho_3 = T_{\mathcal{A}}(t_3, t_2)\rho_2 T_{\mathcal{A}}(t_2, t_3), \tag{26.10}$$

where $T_{\mathcal{A}}(t_3, t_2)$ is the unitary transformation produced by the second beam splitter, (26.3), and we assume that during this process there is no further interaction of the particle with the environment. The result is

$$\rho_3 = \tfrac{1}{2}\left\{(1-\alpha')|e\rangle\langle e| + (1+\alpha')|f\rangle\langle f| + i\alpha''\bigl(|e\rangle\langle f| - |f\rangle\langle e|\bigr)\right\}. \tag{26.11}$$

The diagonal parts of ρ_3, the coefficients of $|e\rangle\langle e|$ and $|f\rangle\langle f|$, are the probabilities that the particle will emerge in the e or the f channel, and are, of course, identical with the expressions in (26.7).

Using a density matrix is particularly convenient for discussing a situation in which the particle interacts with the environment more than once as it passes through the c or the d arm of the interferometer. The simplest situation to analyze is one in which each of these interactions is independent of the others, and they do not alter the wave packet of the particle. In particular, let the environment consist of a number of separate pieces (e.g., separate molecules) with a Hilbert space

$$\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \mathcal{E}_3 \otimes \cdots \mathcal{E}_n \tag{26.12}$$

and an initial state

$$|\epsilon\rangle = |\epsilon_1\rangle \otimes |\epsilon_2\rangle \otimes |\epsilon_3\rangle \otimes \cdots |\epsilon_n\rangle \tag{26.13}$$

at time t_1. The j'th interaction results in $|\epsilon_j\rangle$ changing to $|\epsilon'_j\rangle$ if the particle is in the c arm, or to $|\epsilon''_j\rangle$ if the particle is in the d arm. Thus the net effect of all of these interactions as the particle passes through the interferometer is

$$\begin{aligned} |c\rangle|\epsilon\rangle &= |c\rangle|\epsilon_1\rangle|\epsilon_2\rangle\cdots|\epsilon_n\rangle \mapsto |c'\rangle|\epsilon'\rangle = |c'\rangle|\epsilon'_1\rangle|\epsilon'_2\rangle\cdots|\epsilon'_n\rangle, \\ |d\rangle|\epsilon\rangle &= |d\rangle|\epsilon_1\rangle|\epsilon_2\rangle\cdots|\epsilon_n\rangle \mapsto |d'\rangle|\epsilon''\rangle = |d'\rangle|\epsilon''_1\rangle|\epsilon''_2\rangle\cdots|\epsilon''_n\rangle. \end{aligned} \tag{26.14}$$

The reduced density matrix ρ_2 for the particle just before it passes through the second beam splitter is again of the form (26.8) or (26.9), with

$$\alpha = \langle \epsilon'' | \epsilon' \rangle = \alpha_1 \alpha_2 \cdots \alpha_n, \tag{26.15}$$

where

$$\alpha_j = \langle \epsilon''_j | \epsilon'_j \rangle, \tag{26.16}$$

and ρ_3, when the particle has passed through the second beam splitter, is again given by (26.11).

In a typical situation one would expect the α_j to be less than one, though not necessarily small. Note that if there are a large number of collisions, α in (26.15) can be very small, even if the individual α_j are not themselves small quantities. Thus repeated interactions with the environment will in general lead to greater decoherence than that produced by a single interaction, and if these interactions are of roughly the same kind, one expects the coherence $|\alpha|$ to decrease exponentially with the number of interactions.

Even if the different interactions with the environment are not independent of one another, the net effect may well be much the same, although it might take more interactions to produce a given reduction of $|\alpha|$. In any case, what happens at the second beam splitter, in particular the probability that the particle will emerge in each of the output channels, depends only on the density matrix ρ for the particle when it arrives at this beam splitter, and not on the details of all the scattering processes which have occurred earlier. For this reason, a density matrix is very convenient for analyzing the nature and extent of decoherence in this situation.

26.4 Random Environment

Suppose that the environment which interacts with the particle is itself random, and that at time t_1, when the particle emerges from the first beam splitter, it is described by a density matrix R_1 which can be written in the form

$$R_1 = \sum_j p_j |\epsilon^j\rangle\langle\epsilon^j|, \tag{26.17}$$

where $\{|\epsilon^j\rangle\}$ is an orthonormal basis of $\mathcal{E}$, and $\sum p_j = 1$. Although it is natural, and for many purposes not misleading to think of the environment as being in the state $|\epsilon^j\rangle$ with probability p_j, we shall think of R_1 as simply a pre-probability (Sec. 15.2). Assume that while the particle is inside the interferometer, during the time interval from t_1 to t_2, the interaction with the environment gives rise to unitary transformations

$$|c\rangle|\epsilon^j\rangle \mapsto |c'\rangle|\zeta^j\rangle, \quad |d\rangle|\epsilon^j\rangle \mapsto |d'\rangle|\eta^j\rangle, \tag{26.18}$$

where we are again assuming, as in (26.4) and (26.14), that the environment has a negligible influence on the particle wave packets $|c'\rangle$ and $|d'\rangle$. Because the time evolution is unitary, $\{|\zeta^j\rangle\}$ and $\{|\eta^j\rangle\}$ are orthonormal bases of $\mathcal{E}$.

Let the state of the particle and the environment at t_1 be given by a density matrix

$$\Psi_1 = [\bar{a}] \otimes R_1, \tag{26.19}$$

where $|\bar{a}\rangle = (|c\rangle + |d\rangle)/\sqrt{2}$ is the state of the particle when it emerges from the first beam splitter. At t_2, just before the particle leaves the interferometer, the density matrix resulting from unitary time evolution of the total system will be

$$\begin{aligned}\Psi_2 = \tfrac{1}{2}\sum_j p_j\big\{&|c'\rangle\langle c'| \otimes |\zeta^j\rangle\langle\zeta^j| + |d'\rangle\langle d'| \otimes |\eta^j\rangle\langle\eta^j| \\ &+ |c'\rangle\langle d'| \otimes |\zeta^j\rangle\langle\eta^j| + |d'\rangle\langle c'| \otimes |\eta^j\rangle\langle\zeta^j|\big\}.\end{aligned} \tag{26.20}$$

Taking a partial trace gives the expression

$$\rho_2 = \mathrm{Tr}_{\mathcal{E}}(\Psi_2) = \tfrac{1}{2}\big\{|c'\rangle\langle c'| + |d'\rangle\langle d'| + \alpha|c'\rangle\langle d'| + \alpha^*|d'\rangle\langle c'|\big\}, \tag{26.21}$$

for the reduced density matrix of the particle at time t_2, where

$$\alpha = \sum_j p_j\langle\eta^j|\zeta^j\rangle. \tag{26.22}$$

The expression (26.21) is formally identical to (26.8), but the complex parameter α is now a weighted average of a collection of complex numbers, the inner products $\{\langle\eta^j|\zeta^j\rangle\}$, each with magnitude less than or equal to 1. Consider the case in which

$$\langle\eta^j|\zeta^j\rangle = e^{i\phi_j}, \tag{26.23}$$

that is, the interaction with the environment results in nothing but a phase difference between the wave packets of the particle in the c and d arms. Even though $|\langle\eta^j|\zeta^j\rangle| = 1$ for every j, the sum (26.22) will in general result in $|\alpha| < 1$, and if the sum includes a large number of random phases, $|\alpha|$ can be quite small. Hence a random environment can produce decoherence even in circumstances in which a non-random environment (as discussed in Secs. 26.2 and 26.3) does not.

The basis $\{|\epsilon^j\rangle\}$ in which R_1 is diagonal is useful for calculations, but does not actually enter into the final result for ρ_2. To see this, rewrite (26.18) in the form

$$|c\rangle \otimes |\epsilon\rangle \mapsto |c'\rangle \otimes U_c|\epsilon\rangle, \quad |d\rangle \otimes |\epsilon\rangle \mapsto |d'\rangle \otimes U_d|\epsilon\rangle, \tag{26.24}$$

where $|\epsilon\rangle$ is any state of the environment, and U_c and U_d are unitary transformations on $\mathcal{E}$. Then (26.22) can be written in the form

$$\alpha = \mathrm{Tr}_{\mathcal{E}}\big(R_1 U_d^\dagger U_c\big), \tag{26.25}$$

which makes no reference to the basis $\{|\epsilon^j\rangle\}$.

26.5 Consistency of Histories

Consider a family of histories at times $t_0 < t_1 < t_2 < t_3$ with support

$$\begin{aligned}Y^{ce} &= [\psi_0] \odot [c] \odot [c'] \odot [e],\\ Y^{de} &= [\psi_0] \odot [d] \odot [d'] \odot [e],\\ Y^{cf} &= [\psi_0] \odot [c] \odot [c'] \odot [f],\\ Y^{df} &= [\psi_0] \odot [d] \odot [d'] \odot [f],\end{aligned} \tag{26.26}$$

where $[\psi_0]$ is the initial state $|a\rangle|\epsilon\rangle$ in (26.6), and the unitary dynamics is that of Sec. 26.2. The chain operators for histories which end in $[e]$ are automatically orthogonal to those of histories which end in $[f]$. However, when the final states are the same, the inner products are

$$\langle K(Y^{df}), K(Y^{cf})\rangle = \alpha/4 = -\langle K(Y^{de}), K(Y^{ce})\rangle, \tag{26.27}$$

where α is the parameter defined in (26.5), which appears in the density matrix (26.8) or (26.9). Equation (26.27) is also valid in the case of multiple interactions with the environment, where α is given by (26.15). And it holds for the random environment discussed in Sec. 26.4, with α defined in (26.22), provided one redefines the histories in (26.26) by eliminating the intial state $[\psi_0]$, so that each history begins with $[c]$ or $[d]$ at t_1, and uses the density matrix Ψ_1, (26.19), as an initial state at time t_1 in the consistency condition (15.48). In this case the operator inner product used in (26.27) is $\langle , \rangle_{\Psi_1}$, as it involves the density matrix Ψ_1, see (15.48).

If there is no interaction with the environment, then $\alpha = 1$ and (26.27) implies that the family (26.26) is not consistent. However, if α is very small, even though it is not exactly zero, one can say that the family (26.26) is *approximately consistent*, or consistent for all practical purposes, for the reasons indicated at the end of Sec. 10.2: one expects that by altering the projectors by small amounts one can produce a nearby family which is exactly consistent, and which has essentially the same physical significance as the original family.

This shows that the presence of decoherence may make it possible to discuss the time dependence of a quantum system using a family of histories which in the absence of decoherence would violate the consistency condtions and thus not make sense. This is an important consideration when one wants to understand how the classical behavior of macroscopic objects is consistent with quantum mechanics, which is the topic of the next section.

26.6 Decoherence and Classical Physics

The simple example discussed in the preceding sections of this chapter illustrates two important consequences of decoherence: it can destroy interference effects, and it can render certain families of histories of a subsystem consistent, or at least approximately consistent, when in the absence of decoherence such a family is inconsistent. There is an additional important effect which is not part of decoherence as such, for it can arise in either a classical or a quantum system interacting with its environment: the environment perturbs the motion of the system one is interested in, typically in a random way. (A classical example is Brownian motion, Sec. 8.1.)

The laws of classical mechanics are simple, have an elegant mathematical form, and are quite unlike the laws of quantum mechanics. Nonetheless, physicists believe that classical laws are only an approximation to the more fundamental quantum laws, and that quantum mechanics determines the motion of macroscopic objects made up of many atoms in the same way as it determines the motion of the atoms themselves, and that of the elementary particles of which the atoms are composed. However, showing that classical physics is a limiting case of quantum physics is a non-trivial task which, despite considerable progress, is not yet complete, and a detailed discussion lies outside the scope of this book. The following remarks are intended to give a very rough and qualitative picture of how the correspondence between classical and quantum physics comes about. More detailed treatments will be found in the references listed in the bibliography.

A macroscopic object such as a baseball, or even a grain of dust, is made up of an enormous number of atoms. The description of its motion provided by classical physics ignores most of the mechanical degrees of freedom, and focuses on a rather small number of *collective coordinates*. These are, for example, the center of mass and the Euler angles for a rigid body, to which may be added the vibrational modes for a flexible object. For a fluid, the collective coordinates are the hydrodynamic variables of mass and momentum density, thought of as obtained by "coarse graining" an atomic description by averaging over small volumes which still contain a very large number of atoms. It is important to note that the classical description employs a very special set of quantities, rather than using all the mechanical degrees of freedom.

It is plausible that properties represented by classical collective coordinates, such as "the mass density in region X has the value Y", correspond to projectors onto subspaces of a suitable Hilbert space. These subspaces will have a very large dimension, because the classical description is relatively coarse, and there will not be a unique projector corresponding to a classical property, but instead a collection of projectors (or subspaces), all of which correspond within some approximation to the same classical property.

In the same way, a classical property which changes as a function of time will be associated with different projectors as time progresses, and thus with a quantum history. The continuous time variable of a classical description can be related to the discrete times of a quantum history in much the same way as a continuous classical mass distribution is related to the discrete atoms of a quantum description. Just as a given classical property will not correspond to a unique quantum projector, there will be many quantum histories, and families of histories, which correspond to a given classical description of the motion, and represent it to a fairly good approximation. The term "quasi-classical" is used for such a quantum family and the histories which it contains.

In order for a quasi-classical family to qualify as a genuine quantum description, it must satisfy the consistency conditions. Can one be sure that this is the case? Gell-Mann, Hartle, Brun, and Omnès (see references in the bibliography) have studied this problem, and concluded that there are some fairly general conditions under which one can expect consistency conditions to be at least approximately satisfied for quasi-classical families of the sort one encounters in hydrodynamics or in the motion of rigid objects. That such quasi-classical families will turn out to be consistent is made plausible by the following consideration. Any system of macroscopic size is constantly in contact with an environment. Even a dust particle deep in interstellar space is bombarded by the cosmic background radiation, and will occasionally collide with atoms or molecules. In addition to an external environment of this sort, macroscopic systems have an internal environment constituted by the degrees of freedom left over when the collective coordinates have been specified. Both the external and the internal environment can contribute to processes of decoherence, and these can make it very hard to observe quantum interference effects. While the absence of interference, which is signaled by the fact that the density matrix of the subsystem is (almost) diagonal in a suitable representation, is not the same thing as the consistency of a suitable family of histories, nonetheless the two are related, as suggested by the example considered earlier in this chapter, where the same parameter α characterizes both the degree of coherence of the particle when it leaves the interferometer, and also the extent to which certain consistency conditions are not fulfilled. The effectiveness of this kind of decoherence is what makes it very difficult to design experiments in which macroscopic objects, even those no bigger than large molecules, exhibit quantum interference.

If a quasi-classical family can be shown to be consistent, will the histories in it obey, at least approximately, classical equations of motion? Again, this is a non-trivial question, and we refer the reader to the references in the bibliography for various studies. For example, Omnès has published a fairly general argument that classical and quantum mechanics give similar results if the quantum projectors correspond (approximately) to a cell in the classical phase space which is not too small and has a fairly regular shape, provided that during the time interval of interest the classical equations of motion do not result in too great a distortion of this cell. This last condition can break down rather quickly in the presence of chaos, a situation in which the motion predicted by the classical equations depends in a very sensitive way upon initial conditions.

Classical equations of motion are deterministic, whereas a quantum description employing histories is stochastic. How can these be reconciled? The answer is that the classical equations are idealizations which in appropriate circumstances work rather well. However, one must expect the motion of any real macroscopic system to show some effects of a random environment. The deterministic equations one usually writes down for classical collective coordinates ignore these environmental effects. The equations can be modified to allow for the effects of the environment by including stochastic noise, but then they are no longer deterministic, and this narrows the gap between classical and quantum descriptions. It is also worth keeping in mind that under appropriate conditions the quantum probability associated with a suitable quasi-classical history of macroscopic events can be very close to one. These considerations would seem to remove any conflict between classical and quantum physics with respect to determinism, especially when one realizes that the classical description must in any case be an approximation to some more accurate quantum description.

In conclusion, even though many details have not been worked out and much remains to be done, there is no reason at present to doubt that the equations of classical mechanics represent an appropriate limit of a more fundamental quantum description based upon a suitable set of consistent histories. Only certain aspects of the motion of macroscopic physical bodies, namely those described by appropriate collective coordinates, are governed by classical laws. These laws provide an approximate description which, while quite adequate for many purposes, will need to be supplemented in some circumstances by adding a certain amount of environmental or quantum noise.