Chapter 25

Hardy's Paradox

25.1 Introduction

Hardy's paradox resembles the Bohm version of the Einstein-Podolsky-Rosen paradox, discussed in Chs. 23 and 24, in that it involves two correlated particles, each of which can be in one of two states. However, Hardy's initial state is chosen in such a way that by following a plausible line of reasoning one arrives at a logical contradiction: something is shown to be true which one knows to be false. This makes this paradox in some respects more paradoxical than the EPR paradox as stated in Sec. 24.1. A paradox of a somewhat similar nature involving three spin-half particles was discovered (or invented) by Greenberger, Horne, and Zeilinger a few years earlier. The basic principles behind this GHZ paradox are very similar to those involved in Hardy's paradox. We shall limit our analysis to Hardy's paradox, as it is a bit simpler, but the same techniques can be used to analyze the GHZ paradox.

Hardy's paradox can be discussed in the language of spin-half particles, but we will follow the original paper, though with some minor modifications, in thinking of it as involving two particles, each of which can move through one of two arms (the two arms are analogous to the two states of a spin-half particle) of an interferometer, as indicated in Fig. 25.1. These are particles without spin, or for which the spin degree of freedom plays no role in the gedanken experiment. The source S at the center of the diagram produces two particles a and b moving to the left and right, respectively, in an initial state

$$|\psi_0\rangle = (|c\bar{c}\rangle + |c\bar{d}\rangle + |d\bar{c}\rangle)/\sqrt{3}. \tag{25.1}$$

Here $|c\bar{d}\rangle$ stands for $|c\rangle \otimes |\bar{d}\rangle$, a state in which particle a is in the c arm of the left interferometer, and particle b in the $\bar{d}$ arm of the interferometer on the right. The other kets are defined in the same way. One can think of the two particles as two photons, but other particles will do just as well. In Hardy's original paper one particle was an electron and the other a positron, and the absence of a $d\bar{d}$ term in (25.1) was due to their meeting and annihilating each other.

Suppose that S produces the state (25.1) at the time t_0. The unitary time development from t_0 to a time t_1, which is before either of the particles passes through the beam splitter at the output of its interferometer, is trivial: each particle remains in the same arm in which it starts out. We shall denote the states at t_1 using the same symbol as at t_0: $|c\rangle$, $|\bar{d}\rangle$, etc. One could change c to c', etc., but this is really not necessary. In this simplified notation the time development operator for

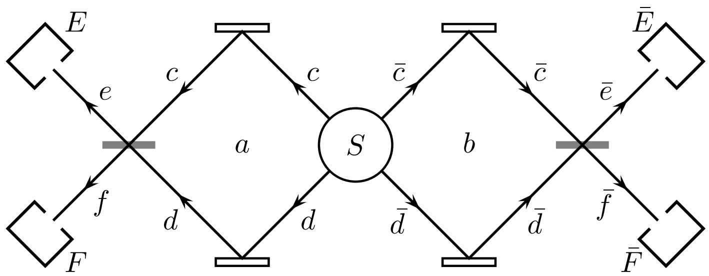

Figure 25.1: Double interferometer for Hardy's paradox.

the time interval from t_0 to t_1 is simply the identity I. During the time interval from t_1 to t_2, each particle passes through the beam splitter at the exit of its interferometer, and these beam splitters produce unitary transformations

$$\begin{aligned} B:\ &|c\rangle \mapsto (|e\rangle + |f\rangle)/\sqrt{2}, \quad |d\rangle \mapsto (-|e\rangle + |f\rangle)/\sqrt{2}, \\ \bar{B}:\ &|\bar{c}\rangle \mapsto (|\bar{e}\rangle + |\bar{f}\rangle)/\sqrt{2}, \quad |\bar{d}\rangle \mapsto (-|\bar{e}\rangle + |\bar{f}\rangle)/\sqrt{2}, \end{aligned} \tag{25.2}$$

where $|e\rangle$, etc., denote wave packets in the output channels, and the phases are chosen to agree with those used for the toy model in Sec. 12.1. Combining the transformations in (25.2) results in the unitary transformations

$$\begin{aligned} |c\bar{c}\rangle &\mapsto (+|e\bar{e}\rangle + |e\bar{f}\rangle + |f\bar{e}\rangle + |f\bar{f}\rangle)/2, \\ |c\bar{d}\rangle &\mapsto (-|e\bar{e}\rangle + |e\bar{f}\rangle - |f\bar{e}\rangle + |f\bar{f}\rangle)/2, \\ |d\bar{c}\rangle &\mapsto (-|e\bar{e}\rangle - |e\bar{f}\rangle + |f\bar{e}\rangle + |f\bar{f}\rangle)/2, \\ |d\bar{d}\rangle &\mapsto (+|e\bar{e}\rangle - |e\bar{f}\rangle - |f\bar{e}\rangle + |f\bar{f}\rangle)/2, \end{aligned} \tag{25.3}$$

for the combined states of the two particles during the time interval from t_0 or t_1 to t_2. Adding up the appropriate terms in (25.3), one finds that the initial state (25.1) is transformed into

$$B\bar{B}:\ |\psi_0\rangle \mapsto \left(-|e\bar{e}\rangle + |e\bar{f}\rangle + |f\bar{e}\rangle + 3|f\bar{f}\rangle\right)/\sqrt{12} \tag{25.4}$$

by the beam splitters.

We will later need to know what happens if one or both of the beam splitters has been taken out of the way. Let O and $\bar{O}$ denote situations in which the left and the right beam splitters, respectively, have been removed. Then (25.2) is to be replaced with

$$\begin{aligned} O:\ &|c\rangle \mapsto |f\rangle, \quad |d\rangle \mapsto |e\rangle, \\ \bar{O}:\ &|\bar{c}\rangle \mapsto |\bar{f}\rangle, \quad |\bar{d}\rangle \mapsto |\bar{e}\rangle, \end{aligned} \tag{25.5}$$

in agreement with what one would expect from Fig. 25.1. The time development of $|\psi_0\rangle$ from t_0 to t_2 if one or both of the beam splitters is absent can be worked out using (25.5) together with (25.2):

$$\begin{aligned} B\bar{O}:\ &|\psi_0\rangle \mapsto \left(|e\bar{e}\rangle + |f\bar{e}\rangle + 2|f\bar{f}\rangle\right)/\sqrt{6}, \\ O\bar{B}:\ &|\psi_0\rangle \mapsto \left(|e\bar{e}\rangle + |e\bar{f}\rangle + 2|f\bar{f}\rangle\right)/\sqrt{6}, \\ O\bar{O}:\ &|\psi_0\rangle \mapsto \left(|e\bar{f}\rangle + |f\bar{e}\rangle + |f\bar{f}\rangle\right)/\sqrt{3}. \end{aligned} \tag{25.6}$$

When they emerge from the beam splitters, the particles are detected, see Fig. 25.1. In order to have a compact notation, we shall use $|M\rangle$ for the initial state of the two detectors for particle a, and $|\bar{M}\rangle$ that of the detectors for particle b, and assume that the process of detection corresponds to the following unitary transformations for the time interval from t_2 to t_3:

$$\begin{aligned} |e\rangle|M\rangle &\mapsto |E\rangle, \quad |f\rangle|M\rangle \mapsto |F\rangle, \\ |\bar{e}\rangle|\bar{M}\rangle &\mapsto |\bar{E}\rangle, \quad |\bar{f}\rangle|\bar{M}\rangle \mapsto |\bar{F}\rangle. \end{aligned} \tag{25.7}$$

Thus $|E\rangle$ means that particle a was detected by the detector located on the e channel. We are now ready to consider the paradox, which can be formulated in two different ways. Both of these are found in Hardy's original paper, though in the opposite order.

25.2 The First Paradox

For this paradox we suppose that both beam splitters are in place. Consider the consistent family of histories at the times t_0, t_1, and t_2 whose support consists of the four histories

$$\mathcal{F}_1 : \ \psi_0 \odot \{\bar{c}, \bar{d}\} \odot \{e, f\}, \tag{25.8}$$

with the same initial state ψ_0, and one of the two possibilities $\bar{c}$ or $\bar{d}$ at t_1, followed by e or f at t_2. Here the symbols stand for projectors associated with the corresponding kets: $\bar{c} = |\bar{c}\rangle\langle\bar{c}|$, etc. That this family is consistent can be seen by noting that the unitary dynamics for particle a is independent of that for particle b at all times after t_0: the time development operator factors. Thus the Heisenberg operators (see Sec. 11.4) for $\bar{c}$ and $\bar{d}$, which refer to the b particle, commute with those for e and f, which refer to the a particle, so that for purposes of checking consistency, (25.8) is the same as a history involving only two times: t_0 and one later time. Hence one can apply the rule that a family of histories involving only two times is automatically consistent, Sec. 11.3. Of course, one can reach the same conclusion by explictly calculating the chain kets (Sec. 11.6) and showing that they are orthogonal to one another.

The history $\psi_0 \odot \bar{c} \odot e$ has zero weight. To see this, construct the chain ket starting with

$$\bar{c}|\psi_0\rangle = (|c\bar{c}\rangle + |d\bar{c}\rangle)/\sqrt{3} = (|c\rangle + |d\rangle) \otimes |\bar{c}\rangle/\sqrt{3}. \tag{25.9}$$

When $T(t_2, t_1)$ is applied to this, the result—see (25.2)—will be $|f\rangle$ times a ket for the b particle, and applying the projector e to it yields zero. As a consequence, since $\psi_0 \odot \bar{d} \odot e$ has finite weight, one has

$$\Pr(\bar{d}, t_1 \,|\, e, t_2) = 1, \tag{25.10}$$

where the times t_1 and t_2 associated with the events $\bar{d}$ and e are indicated explicitly, rather than by subscripts as in earlier chapters. Thus if particle a emerges in e at time t_2, one can be sure that particle b was in the $\bar{d}$ arm at time t_1.

A similar result is obtained if instead of (25.8) one uses the family

$$\mathcal{F}_1' : \ \Psi_0 \odot \{\bar{c}, \bar{d}\} \odot \{E, F\}, \tag{25.11}$$

with events at times t_0, t_1, and t_3, where

$$|\Psi_0\rangle = |\psi_0\rangle \otimes |M\bar{M}\rangle \tag{25.12}$$

includes the initial states of the measuring devices. The fact that $\Psi_0 \odot \bar{c} \odot E$ has zero weight implies that

$$\Pr(\bar{d}, t_1 \,|\, E, t_3) = 1. \tag{25.13}$$

Of course, (25.13) is what one would expect, given (25.10), and vice versa: the measuring device shows that particle a emerged in the e channel if and only if this was actually the case. In the discussion which follows we will, because it is somewhat simpler, use families of the type $\mathcal{F}_1$ which do not include any measuring devices. But the same sort of argument will work if instead of e, $\bar{e}$, etc. one uses measurement outcomes E, $\bar{E}$, etc.

By symmetry it is clear that the family

$$\mathcal{F}_2: \;\; \psi_0 \odot \{c, d\} \odot \{\bar{e}, \bar{f}\}, \tag{25.14}$$

obtained by interchanging the role of particles a and b in (25.8), is consistent. Since the history $\psi_0 \odot c \odot \bar{e}$ has zero weight, it follows that

$$\Pr(d, t_1 \,|\, \bar{e}, t_2) = 1, \tag{25.15}$$

or, if measurements are included,

$$\Pr(d, t_1 \,|\, \bar{E}, t_3) = 1. \tag{25.16}$$

That is, if particle b emerges in channel $\bar{e}$ (the measurement result is $\bar{E}$), then particle a was earlier in the d and not the c arm of its interferometer.

To complete the paradox, we need two additional families. Using

$$\mathcal{F}_3: \;\; \psi_0 \odot I \odot \{e\bar{e}, e\bar{f}, f\bar{e}, f\bar{f}\}, \tag{25.17}$$

one can, see (25.4), show that

$$\Pr(e\bar{e}, t_2) = 1/12. \tag{25.18}$$

Finally, the family

$$\mathcal{F}_4: \;\; \psi_0 \odot \{c\bar{c}, c\bar{d}, d\bar{c}, d\bar{d}\} \odot I \tag{25.19}$$

yields the result

$$\Pr(d\bar{d}, t_1) = 0, \tag{25.20}$$

because $|d\bar{d}\rangle$ occurs with zero amplitude in $|\psi_0\rangle$, (25.1).

Hardy's paradox can be stated in the following way. Whenever a emerges in the e channel we can be sure, (25.10), that b was earlier in the $\bar{d}$ arm, and whenever b emerges in the $\bar{e}$ channel we can be sure, (25.15), that a was earlier in the d arm. The probability that a will emerge in e at the same time that b emerges in $\bar{e}$ is 1/12, (25.18), and when this happens it must be true that a was earlier in d and b was earlier in $\bar{d}$. But given the initial state $|\psi_0\rangle$, it is impossible for a to be in d at the same time that b is in $\bar{d}$, (25.20), so we have reached a contradiction.

Here is a formal argument using probability theory. First, (25.10) implies that

$$\Pr(\bar{d}, t_1 \,|\, e\bar{e}, t_2) = 1, \tag{25.21}$$

because if a conditional probability is equal to 1, it will also be equal to 1 if the condition is made more restrictive, assuming the new condition has positive probability. In the case at hand the condition e is replaced with $e\bar{e}$, and the latter has a probability of 1/12, (25.18). In the same way,

$$\Pr(d, t_1 \,|\, e\bar{e}, t_2) = 1 \tag{25.22}$$

is a consequence of (25.15). Combining (25.21) and (25.22) leads to

$$\Pr(d\bar{d}, t_1 \,|\, e\bar{e}, t_2) = 1, \tag{25.23}$$

and therefore, in light of (25.18),

$$\Pr(d\bar{d}, t_1) \geq 1/12. \tag{25.24}$$

In Hardy's original paper this version of the paradox was constructed in a somewhat different way. Rather than using conditional probabilities to infer properties at earlier times, Hardy reasoned as follows, employing the version of the gedanken experiment in which there is a final measurement. Suppose that both interferometers are extremely large, so that the difference $t_3 - t_1$ is small compared to the time required for light to travel from the source to one of the beam splitters, or from one beam splitter to the other. (The choice of t_1 in our analysis is somewhat arbitrary, but there is nothing wrong with choosing it to be just before the particles arrive at their respective beam splitters.) In this case there is a moving coordinate system or Lorentz frame in which relativistic effects mean that the detection of particle a in the e channel occurs (in this Lorentz frame) at a time when the b particle is still inside its interferometer. In this case, the inference from E to $\bar{d}$ can be made using wave function collapse, Sec. 18.2. By using a different Lorentz frame in which the b particle is the first to pass through its beam splitter, one can carry out the corresponding inference from $\bar{E}$ to d. Next, Hardy made the assumption that inferences of this sort which are valid in one Lorentz frame are valid in another Lorentz frame, and this justifies the analogs of (25.13) and (25.16). With these results in hand, the rest of the paradox is constructed in the manner indicated earlier, with a few obvious changes, such as replacing $e\bar{e}$ in (25.18) with $E\bar{E}$.

25.3 Analysis of the First Paradox

In order to arrive at the contradiction between (25.24) and (25.20), it is necessary to combine probabilities obtained using four different frameworks, $\mathcal{F}_1$ to $\mathcal{F}_4$ (or their counterparts with the measuring apparatus included). While there is no difficulty doing so in classical physics, in the quantum case one must check that the corresponding frameworks are compatible, i.e., there is a single consistent family which contains all of the histories in $\mathcal{F}_1$ to $\mathcal{F}_4$. However, it turns out that *no two of these frameworks are mutually compatible.*

One way to see this is to note that the family

$$\mathcal{J}_1 : \;\psi_0 \odot \{c, d\} \odot \{e, f\} \tag{25.25}$$

is *inconsistent*, as one can show by working out the chain kets and showing that they are not orthogonal. This inconsistency should come as no surprise in view of the discussion of interference in Ch. 13, since the histories in $\mathcal{J}_1$ contain projectors indicating both which arm of the interferometer particle a is in at time t_1 *and* the channel in which it emerges at t_2. To be sure, the initial condition

$|\psi_0\rangle$ is more complicated than its counterpart in Ch. 13, but it would have to be of a fairly special form in order not to give rise to inconsistencies. (It can be shown that each of the four histories in (25.25) is *intrinsically* inconsistent in the sense that it can never occur in a consistent family, Sec. 11.8.) Similarly, the family

$$\mathcal{J}_2: \ \psi_0 \odot \{\bar{c}, \bar{d}\} \odot \{\bar{e}, \bar{f}\} \tag{25.26}$$

is inconsistent.

A comparison of $\mathcal{F}_1$ and $\mathcal{F}_2$, (25.8) and (25.14), shows that a common refinement will necessarily include all of the histories in $\mathcal{J}_1$, since c and d occur in $\mathcal{F}_2$ at t_1, and e and f in $\mathcal{F}_1$ at t_2. Therefore no common refinement can be a consistent family, and $\mathcal{F}_1$ and $\mathcal{F}_2$ are incompatible. In the same way, with the help of $\mathcal{J}_1$ and $\mathcal{J}_2$ one can show that both $\mathcal{F}_1$ and $\mathcal{F}_2$ are incompatible with $\mathcal{F}_3$ and $\mathcal{F}_4$, and that $\mathcal{F}_3$ is incompatible with $\mathcal{F}_4$. As a consequence of these incompatibilities, the derivation of (25.21) from (25.10) is invalid, as is the corresponding derivation of (25.22) from (25.15).

Although $\mathcal{F}_3$ and $\mathcal{F}_4$ are incompatible, there is a consistent family

$$\mathcal{F}_5: \ \psi_0 \odot \{d\bar{d}, I - d\bar{d}\} \odot \{e\bar{e}, e\bar{f}, f\bar{e}, f\bar{f}\} \tag{25.27}$$

from which one can deduce both (25.18) and (25.20). Consequently, the argument which results in a paradox can be constructed by combining results from only three incompatible families, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_5$, rather than four. But three is still two too many.

It is worth pointing out that the defect we have uncovered in the argument in Sec. 25.2, the violation of consistency conditions, has nothing to do with any sort of mysterious long-range influence by which particle b or a measurement carried out on particle b somehow influences particle a, even when they are far apart. Instead, the basic incompatibility is to be found in the fact that $\mathcal{J}_1$, a family which involves *only* properties of particle a after the initial time t_0, is inconsistent. Thus the paradox arises from ignoring the quantum principles which govern what one can consistently say about the behavior of a *single* particle.

A similar comment applies to Hardy's original version of the paradox, for which he employed different Lorentz frames. Although relativistic quantum theory is outside the scope of this book, it is worth remarking that there is nothing wrong with Hardy's *conclusion* that the measurement outcome E for particle a implies that particle b was in the $\bar{d}$ arm of the interferometer before reaching the beam splitter $\bar{B}$, even if some of the *assumptions* used in his argument, such as wave function collapse and the Lorentz invariance of quantum theory, might be open to dispute. For his conclusion is the same as (25.13), a result obtained by straightforward application of quantum principles, without appealing to wave function collapse or Lorentz invariance. Thus the paradox does not, in and of itself, provide any indication that quantum theory is incompatible with special relativity, or that Lorentz invariance fails to hold in the quantum domain.

25.4 The Second Paradox

In this formulation we assume that both beam splitters are in place, and then make a counterfactual comparison with situations in which one or both of them are absent in order to produce a paradox. In order to model the counterfactuals, we suppose that two quantum coins are connected to servomechanisms in the manner indicated in Sec. 19.4, one for each beam splitter. Depending on the outcome of the coin toss, each servomechanism either leaves the beam splitter in place or removes it at the very last instant before the particle arrives.

Consider a family of histories with support

$$\Phi_0 \odot I \odot \{B\bar{B}, B\bar{O}, O\bar{B}, O\bar{O}\} \odot \{E\bar{E}, E\bar{F}, F\bar{E}, F\bar{F}\} \tag{25.28}$$

at times $t_0 < t_1 < t_2 < t_3$, where t_1 is a time before the quantum coin is tossed, t_2 a time after the toss and after the servomechanisms have done their work, but before the particles reach the beam splitters (if still present), and t_3 a time after the detection of each particle in one of the output channels. Note that the definition of t_2 differs from that used in Sec. 25.3 above. The initial state $|\Phi_0\rangle$ includes the quantum coins, servomechanisms and beam splitters, along with $|\psi_0\rangle$, (25.1), for particles a and b.

Various probabilities can be computed with the help of the unitary transformations given in (25.4), (25.6) and (25.7). For our purposes we need only the following results:

$$\Pr(E\bar{E}, t_3 \,|\, B\bar{B}, t_2) = 1/12, \tag{25.29}$$

$$\Pr(E\bar{F}, t_3 \,|\, B\bar{O}, t_2) = 0, \tag{25.30}$$

$$\Pr(F\bar{E}, t_3 \,|\, O\bar{B}, t_2) = 0, \tag{25.31}$$

$$\Pr(E\bar{E}, t_3 \,|\, O\bar{O}, t_2) = 0. \tag{25.32}$$

These probabilities can be used to construct a counterfactual paradox in the following manner.

H1. Consider a case in which $B\bar{B}$ occurs as a result of the quantum coin tosses, and the outcome of the final measurement on particle a is E.

H2. Suppose that instead of being present, the beam splitter $\bar{B}$ had been absent, $B\bar{O}$. The removal of a distant beam splitter at the last moment could not possibly have affected the outcome of the measurement on particle a, so E would have occurred in case $B\bar{O}$, just as it did in case $B\bar{B}$.

H3. Since by (25.30) $E\bar{F}$ is impossible in this situation, $E\bar{E}$ would have occurred in the case $B\bar{O}$.

H4. Given that $\bar{E}$ would have occurred in the case $B\bar{O}$, it would also have occurred with both beam splitters absent, $O\bar{O}$, since, once again, the removal of a distant beam splitter B at the last instant could not possibly have affected the outcome of a measurement on particle b.

H5. It follows from H1 to H4 that if E occurs in the case $B\bar{B}$, then $\bar{E}$ would have occurred, in this particular experiment, if the quantum coin tosses had resulted in both beam splitters being absent, $O\bar{O}$, rather than present.

H6. Upon interchanging the roles of particles a and b in H1 through H4, we conclude that if $\bar{E}$ occurs in the case $B\bar{B}$, then E would have been the case had both beam splitters been absent, $O\bar{O}$.

H7. Consider a situation in which both E and $\bar{E}$ occur in the case $B\bar{B}$; note that the probability for this is greater than zero, (25.29). Then in the counterfactual situation in which $O\bar{O}$ was the case rather than $B\bar{B}$, we can conclude using H5 that $\bar{E}$, and using H6 that E would have occurred. That is, the outcome of the measurements would have been $E\bar{E}$ had the quantum coin tosses resulted in $O\bar{O}$.

H8. But according to (25.32), $E\bar{E}$ cannot occur in the case $O\bar{O}$, so we have reached a contradiction.

25.5 Analysis of the Second Paradox

A detailed analysis of H1 to H4 is a bit complicated, since both H2 and H4 involve counterfactuals, and the conclusion, stated in H5, comes from chaining together two counterfactual arguments. In order not to become lost in intricate details of how one counterfactual may be combined with another, it is best to focus on the end result in H5, which can be restated in the following way: If in the actual world the quantum coin tosses result in $B\bar{B}$ and the measurement outcome is E, then in a counterfactual world in which the coin tosses had resulted in $O\bar{O}$, particle b would have triggered detector $\bar{E}$.

To support this argument using the scheme of counterfactual reasoning discussed in Sec. 19.4, we need to specify a *single* consistent family which contains the events we are interested in, which are the outcomes of the coin tosses and at least some of the outcomes of the final measurements, together with some event (or perhaps events) at a time earlier than when the quantum coins were tossed, which can serve as a suitable pivot. The framework might contain more than this, but it must contain at least this much. (Note that the pivot event or events can make reference to both particles, and could be more complicated than simply the product of a projector for a times a projector for b.) From this point of view, the intermediate steps in the argument—for example, H2, in which only one of the beam splitters is removed—can be thought of as a method for finding the final framework and pivot through a series of intermediate steps. That is, we may be able to find a framework and pivot which will justify H2, and then modify the framework and choose another pivot, if necessary, in order to incorporate H3 and H4, so as to arrive at the desired result in H5.

We shall actually follow a somewhat different procedure: make a guess for a framework which will support the result in H5, and then check that it works. An intelligent guess is not difficult, for E in case $B\bar{B}$ implies that the b particle was earlier in the $\bar{d}$ arm of its interferometer, (25.13), and when the beam splitter $\bar{B}$ is out of the way, a particle in $\bar{d}$ emerges in the $\bar{e}$ channel, which will result in $\bar{E}$. This suggests taking a look at the consistent family containing the following histories, in which the alternatives $\bar{c}$ and $\bar{d}$ occur at t_1:

$$\Phi_0 \odot \begin{cases} \bar{c} \odot \begin{cases} B\bar{B} \odot F, \\ O\bar{O} \odot \bar{F}, \end{cases} \\ \bar{d} \odot \begin{cases} B\bar{B} \odot \{E, F\}, \\ O\bar{O} \odot \bar{E}. \end{cases} \end{cases} \tag{25.33}$$

The $B\bar{O}$ and $O\bar{B}$ branches have been omitted from (25.33) in order to save space and allow us to concentrate on the essential task of finding a counterfactual argument which leads from $B\bar{B}$ to $O\bar{O}$. Including these other branches terminated by a noncommittal I at t_3 will turn (25.33) into the support of a consistent family without having any effect on the following argument.

The consistency of (25.33) can be seen in the following way. The events $B\bar{B}$ and $O\bar{O}$ are macroscopically distinct, hence orthogonal, and since they remain unchanged from t_2 to t_3, we only need to check that the chain operators for the two histories involving $O\bar{O}$ are orthogonal to each other—as is obviously the case, since the final $\bar{E}$ and $\bar{F}$ are orthogonal—and the chain operators for the three histories involving $B\bar{B}$ are mutually orthogonal. The only conceivable problem arises because two of the $B\bar{B}$ histories terminate with the same projector F. However, because at earlier times these histories involve orthogonal states $\bar{c}$ and $\bar{d}$ of particle b, and F has to do with particle

a (i.e., a measurement on particle a), rather than b, the chain operators are, indeed, orthogonal. The reader can check this by working out the chain kets.

One can use (25.33) to support the conclusion of H5 in the following way. The outcome E in the case $B\bar{B}$ occurs in only one history, on the third line in (25.33). Upon tracing this outcome back to $\bar{d}$ as a pivot, and then moving forward in time on the $O\bar{O}$ branch we come to $\bar{E}$ as the counterfactual conclusion. Having obtained the result in H5, we do not need to discuss H2, H3, and H4. However, it is possible to justify these statements as well by adding a $B\bar{O}$ branch to (25.33) with suitable measurement outcomes at t_3 in place of the noncommittal I, and then adding some additional events involving properties of particle a at time t_1 in order to construct a suitable pivot for the argument in H2. As the details are not essential for the present discussion, we leave them as a (nontrivial) exercise for anyone who wishes to explore the argument in more depth.

By symmetry, H6 can be justified by the use of a consistent family (with, once again, the $B\bar{O}$ and $O\bar{B}$ branches omitted)

$$\Phi_0 \odot \begin{cases} c \odot \begin{cases} B\bar{B} \odot \bar{F}, \\ O\bar{O} \odot F, \end{cases} \\ d \odot \begin{cases} B\bar{B} \odot \{\bar{E}, \bar{F}\}, \\ O\bar{O} \odot E, \end{cases} \end{cases} \tag{25.34}$$

which is (25.33) with the roles of a and b interchanged. However, H7, which combines the results of H5 and H6, is not valid, because the family (25.33) on which H5 is based is incompatible with the family (25.34) on which H6 is based. The problem with combining these two families is that when one introduces the events E and F at t_3 in the $B\bar{B}$ branch of a family which contains c and d at an earlier time, it is essentially the same thing as introducing e and f to make the inconsistent family $\mathcal{J}_1$, (25.25). In the same way, introducing $\bar{E}$ and $\bar{F}$ in the $B\bar{B}$ branch following an earlier $\bar{c}$ and $\bar{d}$ leads to trouble. Even the very first statement in H7, that $E\bar{E}$ occurs in case $B\bar{B}$ with a positive probability, requires the use of a family which is incompatible with both (25.33) and (25.34)! Thus the road to a contradiction is blocked by the single framework rule.

This procedure for blocking the second form of Hardy's paradox is very similar to the one used in Sec. 25.3 for blocking the first form of the paradox. Indeed, for the case $B\bar{B}$ we have used essentially the same families; the only difference comes from the (somewhat arbitrary) decision to word the second form of the paradox in terms of measurement outcomes, and the first in a way which only makes reference to particle properties.

The second form of Hardy's paradox, like the first, cannot be used to justify some form of quantum nonlocality in the sense of some mysterious long-range influence of the presence or absence of a beam splitter in the path of one particle on the behavior of the other particle. Locality was invoked in H2 and H4 (and at the corresponding points in H6). But H2 and H4, as well as the overall conclusion in H5 can be supported by using a suitable framework and pivots. (We have only given the explicit argument for H5.) Thus, while our analysis does not prove that the locality assumptions entering H2 and H4 are correct, it shows that there is no reason to suspect that there is anything wrong with them. The overall argument, H1-H8, results in a contradiction. However, the problem lies not in the locality assumptions in the earlier statements, but rather in the quantum incompatibility overlooked when writing down the otherwise plausible H7. This incompatibility, as noted earlier, has to do with the way a single particle is being described, so it cannot be blamed on anything nonlocal.

Our analysis of H1 through H6 was based upon particular frameworks. As there are a large number of different possible frameworks, one might suppose that an alternative choice might be able to support the counterfactual arguments and lead to a contradiction. There is, however, a relatively straightforward argument to demonstrate that no single framework, and thus no set of compatible frameworks, could possibly support the argument in H1-H7. Consider any framework which contains $E\bar{E}$ at t_3 both in the case $B\bar{B}$ and also in the case $O\bar{O}$. In this framework both (25.29) and (25.32) are valid: $E\bar{E}$ occurs with finite probability in case $B\bar{B}$, and with zero probability in case $O\bar{O}$. The reason is that even though (25.29) and (25.32) were obtained using the framework (25.28), it is a general principle of quantum reasoning, see Sec. 16.3, that the probability assigned to a collection of events in one framework will be precisely the same in *all* frameworks which contain these events and the same initial data (Φ_0 in the case at hand). But in any single framework in which $E\bar{E}$ occurs with probability zero in the case $O\bar{O}$ it is clearly impossible to reach the conclusion at the end of a series of counterfactual arguments that $E\bar{E}$ would have occurred with both beam splitters absent had the outcomes of the quantum coin tosses been different from what actually occurred.

To be more specific, suppose one could find a framework containing a pivot P at t_1 with the following properties: (i) P must have occurred if $B\bar{B}$ was followed by $E\bar{E}$; (ii) if P occurred and was then followed by $O\bar{O}$, the measurement outcome would have been $E\bar{E}$. These are the properties which would permit this framework to support the counterfactual argument in H1-H7. But since $B\bar{B}$ followed by $E\bar{E}$ has a positive probability, the same must be true of P, and therefore $O\bar{O}$ followed by $E\bar{E}$ would also have to occur with a finite probability. (A more detailed analysis shows that $\Pr(E\bar{E}, t_3 \,|\, O\bar{O}, t_2)$ would have to be at least as large as $\Pr(E\bar{E}, t_3 \,|\, B\bar{B}, t_2)$.) However, since $O\bar{O}$ is, in fact, never followed by $E\bar{E}$, a framework and pivot of this kind does not exist.

The conclusion is that it is impossible to use quantum reasoning in a consistent way to arrive at the conclusion H7 starting from the assumption H1. In some respects the analysis just presented seems too simple: it says, in effect, that if a counterfactual argument of the form H1-H7 arrives at a contradiction, then this very fact means there is some way in which this argument violates the rules of quantum reasoning. Can one dispose of a (purported) paradox in such a summary fashion? Yes, one can. The rule requiring that quantum reasoning of this type employ a *single* framework means that the usual rules of ordinary (classical) reasoning and probability theory can be applied as long as one sticks to this particular framework, and there can be no contradiction. To put the matter in a different way, if there is some very clever way to produce this paradox *using only one framework*, then there will also be a corresponding "classical" paradox, and whatever it is that is paradoxical will not be unique to quantum theory.

Nonetheless, there is some value in our working out specific aspects of the paradox using the explicit families (25.33) and (25.34), for they indicate that the basic difficulty with the argument in H1 to H8 lies in an implicit assumption that the different frameworks are compatible, an assumption which is easy to make because it is always valid in classical mechanics. Incompatibility rather than some mysterious nonlocality is the crucial feature which distinguishes quantum from classical physics, and ignoring it is what has led to a paradox.