Chapter 22

Incompatibility Paradoxes

22.1 Simultaneous Values

There is never any difficulty in supposing that a classical mechanical system possesses, at a particular instant of time, precise values of each of its physical variables: momentum, kinetic energy, potential energy of interaction between particles 3 and 5, etc. Physical variables, see Sec. 5.5, correspond to real-valued functions on the classical phase space, and if at some time the system is described by a point γ in this space, the variable A has the value $A(\gamma)$, B has the value $B(\gamma)$, etc.

In quantum theory, where physical variables correspond to observables, that is, Hermitian operators on the Hilbert space, the situation is very different. As discussed in Sec. 5.5, a physical variable A has the value a_j provided the quantum system is in an eigenstate of A with eigenvalue a_j or, more generally, if the system has a property represented by a non-zero projector P such that

$$AP = a_j P. \tag{22.1}$$

It is very often the case that two quantum observables have no eigenvectors in common, and in this situation it is impossible to assign values to both of them for a single quantum system at a single instant of time. This is the case for S_x and S_z for a spin-half particle, and as was pointed out in Sec. 4.6, even the assumption that "$S_z = 1/2$ AND $S_x = 1/2$" is a false (rather than a meaningless) statement is enough to generate a paradox if one uses the usual rules of classical logic. This is perhaps the simplest example of an *incompatibility paradox* arising out of the assumption that quantum properties behave in much the same way as classical properties, so that one can ignore the rules of quantum reasoning summarized in Ch. 16, in particular the rule which forbids combining incompatible properties and families. By contrast, if two Hermitian operators A and B commute, there is at least one orthonormal basis $\{|j\rangle\}$, Sec. 3.7, in which both are diagonal,

$$A = \sum_j a_j |j\rangle\langle j|, \quad B = \sum_j b_j |j\rangle\langle j|. \tag{22.2}$$

If the quantum system is described by this framework, there is no difficulty with supposing that A has (to take an example) the value a_2 at the same time as B has the value b_2.

The idea that *all* quantum variables should simultaneously possess values, as in classical mechanics, has a certain intuitive appeal, and one can ask whether there is not some way to extend the usual Hilbert space description of quantum mechanics, perhaps by the addition of some hidden variables, in order to allow for this possibility. For this to be an extension rather than a completely new theory, one needs to place some restrictions upon which values will be allowed, and the following are reasonable requirements: (i) The value assigned to a particular observable will always be one of its eigenvalues. (ii) Given a collection of *commuting observables*, the values assigned to them will be eigenvalues corresponding to a single eigenvector. For example, with reference to A and B in (22.2), assigning a_2 to A and b_2 to B is a possibility, but assigning a_2 to A and b_3 to B (assuming $a_2 \neq a_3$ and $b_2 \neq b_3$) is not. That condition (ii) is reasonable if one intends to assign values to *all* observables can be seen by noting that the projector $|2\rangle\langle 2|$ in (22.2) is itself an observable with eigenvalues 0 and 1. If it is assigned the value 1, then it seems plausible that A should be assigned the value a_2 and B the value b_2.

Bell and Kochen and Specker have shown that in a Hilbert space of dimension 3 or more, assigning values to all quantum observables in accordance with (i) and (ii) is not possible. In Sec. 22.3 we shall present a simple example due to Mermin which shows that such a value assignment is not possible in a Hilbert space of dimension 4 or more. Such a counterexample is a paradox in the sense that it represents a situation that is surprising and counterintuitive from the perspective of classical physics. Section 22.2 is devoted to introducing the notion of a value functional, a concept which is useful for discussing the two-spin paradox of Sec. 22.3. A truth functional, Sec. 22.4, is a special case of a value functional, and is useful for understanding how the concept of "truth" is used in quantum descriptions. The three-box paradox in Sec. 22.5 employs incompatible frameworks of histories in a manner similar to the way in which the two-spin paradox uses incompatible frameworks of properties at one time, and Sec. 22.6 extends the results of Sec. 22.4 on truth functionals to the case of histories.

22.2 Value Functionals

A *value functional* v assigns to all members $A, B, \dots$ of some collection $\mathcal{C}$ of physical variables numerical values of a sort which could be appropriate for describing a single system at a single instant of time. For example, with γ a fixed point in the classical phase space, the value functional v_γ assigns to each physical variable C the value

$$v_\gamma(C) = C(\gamma) \tag{22.3}$$

of the corresponding function at the point γ. In this case $\mathcal{C}$ could be the collection of all physical variables, or some more restricted set. If there is some algebraic relationship among certain physical variables, as in the formula

$$E = p^2/2m + V \tag{22.4}$$

for the total energy in terms of the momentum and potential energy of a particle in one dimension, this relationship will also be satisfied by the values assigned by v_γ:

$$v_\gamma(E) = [v_\gamma(p)]^2/2m + v_\gamma(V). \tag{22.5}$$

To define a value functional for a quantum system, let $\{D_j\}$ be some fixed decomposition of the identity, and let the collection $\mathcal{C}$ consist of all operators of the form

$$C = \sum_j c_j D_j, \tag{22.6}$$

with real eigenvalues c_j. The value functional v_k defined by

$$v_k(C) = c_k, \tag{22.7}$$

assigns to each physical variable C its value on the subspace D_k. Note that there are as many distinct value functionals as there are members in the decomposition $\{D_j\}$. As in the classical case, if there is some algebraic relationship among the observables belonging to $\mathcal{C}$, such as

$$F = 2I - A + B^2, \tag{22.8}$$

it will be reflected in the values assigned by v_k:

$$v_k(F) = 2 - v_k(A) + [v_k(B)]^2. \tag{22.9}$$

It is important to note that the class $\mathcal{C}$ on which a quantum value functional is defined is a collection of *commuting* observables, since the decomposition of the identity is held fixed and only the eigenvalues in (22.6) are allowed to vary. Conversely, given a collection of commuting observables, one can find an orthonormal basis in which they are simultaneously diagonal, Sec. 3.7, and the corresponding decomposition of the identity can be used to define value functionals which assign values simultaneously to all of the observables in the collection.

The problem posed in Sec. 22.1 of defining values for *all* quantum observables can be formulated as follows: Find a *universal value functional* v_u defined on the collection of *all* observables or Hermitian operators on a quantum Hilbert space, and satisfying the conditions:

U1. For any observable A, $v_u(A)$ is one of its eigenvalues.

U2. Given any decomposition of the identity $\{D_j\}$, with $\mathcal{C}$ the corresponding collection of observables of the form (22.6), there is some D_k from the decomposition such that

$$v_u(C) = c_k \tag{22.10}$$

for every C in $\mathcal{C}$, where c_k is the coefficient in (22.6).

Conditions U1 and U2 are the counterparts of the requirements (i) and (ii) stated in Sec. 22.1. Note that any algebraic relationship, such as (22.8), among the members of a collection of *commuting* observables will be reflected in the values assigned to them by v_u, as in (22.9). The reason is that there will be an orthonormal basis in which these observables are simultaneously diagonal, and (22.10) will hold for the corresponding decomposition of the identity.

22.3 Paradox of Two Spins

There are various examples which show explicitly that a universal value functional satisfying conditions U1 and U2 in Sec. 22.2 cannot exist. One of the simplest is the following two-spin paradox due to Mermin. For a spin-half particle let σ_x, σ_y, and σ_z be the operators $2S_x$, $2S_y$, and $2S_z$, with eigenvalues ± 1. The corresponding matrices using a basis of $|z^+\rangle$ and $|z^-\rangle$ are the familiar Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{22.11}$$

The Hilbert space for two spin-half particles a and b is the tensor product

$$\mathcal{H} = \mathcal{A} \otimes \mathcal{B}, \tag{22.12}$$

and we define the corresponding spin operators as

$$\sigma_{ax} = \sigma_x \otimes I, \quad \sigma_{by} = I \otimes \sigma_y, \tag{22.13}$$

etc.

The nine operators on $\mathcal{H}$ in the 3×3 square

$$\begin{array}{ccc} \sigma_{ax} & \sigma_{bx} & \sigma_{ax}\sigma_{bx} \\ \sigma_{by} & \sigma_{ay} & \sigma_{ay}\sigma_{by} \\ \sigma_{ax}\sigma_{by} & \sigma_{ay}\sigma_{bx} & \sigma_{az}\sigma_{bz} \end{array} \tag{22.14}$$

have the following properties:

M1. Each operator is Hermitian, with two eigenvalues equal to $+1$ and two equal to -1.

M2. The three operators in each row commute with each other, and likewise the three operators in each column.

M3. The product of the three operators in each row is equal to the identity I.

M4. The product of the three operators in both of the first two columns is I, while the product of those in the last column is $-I$.

These statements can be verified by using the well-known properties of the Pauli matrices:

$$(\sigma_x)^2 = I, \quad \sigma_x \sigma_y = i\sigma_z, \tag{22.15}$$

etc. Note that σ_{ax} and σ_{by} commute with each other, as they are defined on separate factors in the tensor product, see (22.13), whereas σ_{ax} and σ_{ay} do not commute with each other. Statement M1 is obvious when one notes that the trace of each of the nine operators in (22.14) is 0, whereas its square is equal to I.

A universal value functional v_u will assign one of its eigenvalues, $+1$ or -1, to each of the nine observables in (22.14). Since the product of the operators in the first row is I and an assignment of values preserves algebraic relations among commuting observables, as in (22.9), it must be the case that

$$v_u(\sigma_{ax})\, v_u(\sigma_{bx})\, v_u(\sigma_{ax}\sigma_{bx}) = 1. \tag{22.16}$$

The products of the values in the other rows and in the first two columns is also 1, whereas for the last column the product is

$$v_u(\sigma_{ax}\sigma_{bx})\, v_u(\sigma_{ay}\sigma_{by})\, v_u(\sigma_{az}\sigma_{bz}) = -1. \tag{22.17}$$

The set of values

$$\begin{array}{ccc} -1 & -1 & +1 \\ +1 & -1 & -1 \\ -1 & -1 & +1 \end{array} \tag{22.18}$$

for the nine observables in (22.14) satisfies (22.16), (22.17), and all of the other product conditions except that the product of the integers in the center column is -1 rather than $+1$. This seems like a small defect, but there is no obvious way to remedy it, since changing any -1 in this column to $+1$ will result in a violation of the product condition for the corresponding row. In fact, a value assignment simultaneously satisfying all six product conditions is impossible, because the three product conditions for the rows imply that the product of all nine numbers is $+1$, while the three product conditions for the columns imply that this same product must be -1, an obvious contradiction. To be sure, we have only looked at a rather special collection of observables in (22.14), but this is enough to show that there is no *universal* value functional capable of assigning values to *every* observable in a manner which satisfies conditions U1 and U2 of Sec. 22.2.

It should be emphasized that the two-spin paradox is not a paradox for quantum mechanics as such, because quantum theory provides no mechanism for assigning values simultaneously to non-commuting observables (except for special cases in which they happen to have a common eigenvector). Each of the nine observables in (22.14) commutes with four others: two in the same row, and two in the same column. However, it does *not* commute with the other four observables. Hence there is no reason to expect that a single value functional can assign sensible values to all nine, and indeed it cannot. The motivation for thinking that such a function might exist comes from the analogy provided by classical mechanics, as noted in Sec. 22.1. What the two-spin paradox shows is that at least in this respect there is a profound difference between quantum and classical physics.

This example shows that a universal value functional is not possible in a four-dimensional Hilbert space, or in any Hilbert space of higher dimension, since one could set up the same example in a four-dimensional subspace of the larger space. The simplest known examples showing that universal value functionals are impossible in a three-dimensional Hilbert space are much more complicated. Universal value functionals are possible in a two-dimensional Hilbert space, a fact of no particular physical significance, since very little quantum theory can be carried out if one is limited to such a space.

22.4 Truth Functionals

Additional insight into the difference between classical and quantum physics comes from considering *truth functionals*. A truth functional is a value functional defined on a collection of indicators (in the classical case) or projectors (in the quantum case), rather than on a more general collection of physical variables or observables. A classical truth functional θ_γ can be defined by choosing a fixed point γ in the phase space, and then for every indicator P belonging to some collection $\mathcal{L}$ writing

$$\theta_\gamma(P) = P(\gamma), \tag{22.19}$$

which is the same as (22.3). Since an indicator can only take the values 0 or 1, $\theta_\gamma(P)$ will either be 1, signifying that system in the state γ possesses the property P, and thus that P is *true*; or 0, indicating that P is *false*. (Recall that the indicator for a classical property, (4.1), takes the value 1 on the set of points in the phase space where the system has this property, and 0 elsewhere.)

A quantum truth functional is defined on a Boolean algebra $\mathcal{L}$ of projectors of the type

$$P = \sum_j \pi_j D_j, \tag{22.20}$$

where each π_j is either 0 or 1, and $\{D_j\}$ is a decomposition of the identity. It has the form

$$\theta_k(P) = \begin{cases} 1 & \text{if} \quad PD_k = D_k, \\ 0 & \text{if} \quad PD_k = 0, \end{cases} \tag{22.21}$$

for some choice of k. This is a special case of (22.7), with (22.20) and π_k playing the role of (22.6) and c_k. If one thinks of the decomposition $\{D_j\}$ as a sample space of mutually exclusive events, one and only one of which occurs, then the truth functional θ_k assigns the value 1 to all properties P which are true, in the sense that $\Pr(P \,|\, D_k) = 1$, when D_k is the event which actually occurs, and 0 to all properties P which are false, in the sense that $\Pr(P \,|\, D_k) = 0$. Thus as long as one only considers a single decomposition of the identity the situation is analogous to the classical case: the projectors in $\{D_j\}$ constitute what is, in effect, a discrete phase space. The difference between classical and quantum physics lies in the fact that θ_γ in (22.19) can be applied to as large a collection of indicators as one pleases, whereas the definition θ_k in (22.21) will not work for an arbitrary collection of projectors; in particular, if P does not commute with D_k, PD_k is not a projector.

For a given decomposition $\{D_j\}$ of the identity, the truth functional θ_k is simply the value functional v_k of (22.7) restricted to projectors belonging to $\mathcal{L}$ rather than to more general operators; that is,

$$\theta_k(P) = v_k(P) \tag{22.22}$$

for all P in $\mathcal{L}$. Conversely, v_k is determined by θ_k in the sense that for any operator C of the form (22.6) one has

$$v_k(C) = \sum_j c_j \theta_k(D_j). \tag{22.23}$$

An alternative approach to defining a truth functional is the following. Let $\theta(P)$ assign the value 0 or 1 to every projector in the Boolean algebra $\mathcal{L}$ generated by the decomposition of the identity $\{D_j\}$, subject to the following conditions:

$$\begin{aligned} \theta(I) &= 1, \\ \theta(I - P) &= 1 - \theta(P), \\ \theta(PQ) &= \theta(P)\theta(Q). \end{aligned} \tag{22.24}$$

One can think of these as a special case of a value functional preserving algebraic relations, as discussed in Sec. 22.2. Thus it is evident that θ_k as defined in (22.21), since it is derived from a value functional, (22.22), will satisfy (22.24). It can also be shown that a functional θ taking the values 0 and 1 and satisfying (22.24) must be of the form (22.21) for some k.

We shall define a *universal truth functional* to be a functional θ_u which assigns 0 or 1 to *every* projector P on the Hilbert space, not simply those associated with a particular Boolean algebra $\mathcal{L}$, in such a way that the relations in (22.24) are satisfied whenever they make sense. In particular, the third relation in (22.24) makes no sense if P and Q do not commute, for then PQ is not a projector, so we modify it to read:

$$\theta_u(PQ) = \theta_u(P)\theta_u(Q) \quad \text{if} \quad PQ = QP. \tag{22.25}$$

When P and Q both belong to the same Boolean algebra they commute with each other, so θ_u when restricted to a particular Boolean algebra $\mathcal{L}$ satisfies (22.24). Consequently, when θ_u is thought of as a function on the projectors in $\mathcal{L}$, it coincides with an "ordinary" truth functional θ_k for this algebra, for some choice of k.

Given a universal value functional v_u, we can define a corresponding universal truth functional θ_u by letting $\theta_u(P) = v_u(P)$ for every projector P. Conversely, given a universal truth functional one can use it to construct a universal value functional satisfying conditions U1 and U2 of Sec. 22.2 by using the counterpart of (22.23):

$$v_u(C) = \sum_j c_j \theta_u(D_j). \tag{22.26}$$

That is, given any Hermitian operator C there is a decomposition of the identity $\{D_j\}$ such that C can be written in the form (22.6). On this decomposition of the identity θ_u must agree with θ_k for some k, so the right side of (22.23) makes sense, and can be used to define $v_u(C)$. It then follows that U1 and U2 of Sec. 22.2 are satisfied. If the eigenvalues of C are degenerate there is more than one way of writing it in the form (22.6), but it can be shown that the properties we are assuming for θ_u imply that these different possibilities lead to the same $v_u(C)$.

This close connection between universal value functionals and universal truth functionals means that arguments for the existence or nonexistence of one immediately apply to the other. Thus neither of these universal functionals can be constructed in a Hilbert space of dimension 3 or more, and the two-spin paradox of Sec. 22.3, while formulated in terms of a universal value functional, also demonstrates the nonexistence of a universal truth functional in a four (or higher) dimensional Hilbert space. It is, indeed, somewhat disappointing that there is nothing very significant to which the formulas (22.25) and (22.26) actually apply!

The nonexistence of universal quantum truth functionals is not very surprising. It is simply another manifestation of the fact that quantum incompatibility makes it impossible to extend certain ideas associated with the classical notion of truth into the quantum domain. Similar problems were discussed earlier in Sec. 4.6 in connection with incompatible properties, and in Sec. 16.4 in connection with incompatible frameworks.

22.5 Paradox of Three Boxes

The three-box paradox of Aharonov and Vaidman resembles the two-spin paradox of Sec. 22.3 in that it is a relatively simple example which is incompatible with the existence of a universal truth functional. Whereas the two-spin paradox refers to properties of a quantum system at a single instant of time, the three-box paradox employs histories, and the incompatibility of the different frameworks reflects a violation of consistency conditions rather than the fact that projectors do not commute with each other. The paradox is discussed in this section, and the connection with truth functionals for histories is worked out in Sec. 22.6.

Consider a three-dimensional Hilbert space spanned by an orthonormal basis consisting of three states $|A\rangle$, $|B\rangle$, and $|C\rangle$. As in the original statement of the paradox, we shall think of these states as corresponding to a particle being in one of three separate boxes, though one could equally well suppose that they are three orthogonal states of a spin-one particle, or the states $m = -1$, 0, and 1 in a toy model of the type introduced in Sec. 2.5. The dynamics is trivial, $T(t', t) = I$: if the particle is in one of the boxes, it stays there. We shall be interested in quantum histories involving three times $t_0 < t_1 < t_2$, based upon an initial state

$$|D\rangle = \big(|A\rangle + |B\rangle + |C\rangle\big)/\sqrt{3} \tag{22.27}$$

at t_0, and ending at t_2 in one of the two events F or $\tilde{F} = I - F$, where F is the projector corresponding to

$$|F\rangle = \big(|A\rangle + |B\rangle - |C\rangle\big)/\sqrt{3}. \tag{22.28}$$

In the first consistent family $\mathcal{A}$ the events at the intermediate time t_1 are A and $\tilde{A} = I - A$, with A the projector $|A\rangle\langle A|$. The support of this family consists of the three histories

$$D \odot \begin{cases} A \odot F, \\ A \odot \tilde{F}, \\ \tilde{A} \odot \tilde{F}, \end{cases} \tag{22.29}$$

since $D \odot \tilde{A} \odot F$ has zero weight. Checking consistency is straightforward. The chain operator for the first history in (22.29) is obviously orthogonal to the other two because of the final states. The orthogonality of the chain operators for the second and third histories can be worked out using chain kets, or by replacing the final $\tilde{F}$ with F and employing the trick discussed in connection with (11.5). Because $D \odot \tilde{A} \odot F$ has zero weight, if the event F occurs at t_2, then A rather than $\tilde{A}$ must have been the case at t_1; i.e., $\tilde{A}$ at t_1 is never followed by F at t_2. Thus one has

$$\Pr(A_1 \,|\, D_0 \wedge F_2) = 1, \tag{22.30}$$

with our usual convention of a subscript indicating the time of an event.

Now consider a second consistent family $\mathcal{B}$ with events B and $\tilde{B} = I - B$ at t_1; B is the projector $|B\rangle\langle B|$. In this case the support consists of the histories

$$D \odot \begin{cases} B \odot F, \\ B \odot \tilde{F}, \\ \tilde{B} \odot \tilde{F}, \end{cases} \tag{22.31}$$

from which one can deduce that

$$\Pr(B_1 \,|\, D_0 \wedge F_2) = 1, \tag{22.32}$$

the obvious counterpart of (22.30) given the symmetry between $|A\rangle$ and $|B\rangle$ in the definition of $|D\rangle$ and $|F\rangle$.

The paradox arises from noting that from the same initial data D and F ("initial" refers to position in a logical argument, not temporal order in a history; see Sec. 16.1) one is able to infer by using $\mathcal{A}$ that A occurred at time t_1, and by using $\mathcal{B}$ that B was the case at t_1. However, A and B are mutually exclusive properties, since $BA = 0$. That is, we seem to be able to conclude with probability one that the particle was in box A, and also that it was in box B, despite the fact that the rules of quantum theory indicate that it cannot simultaneously be in both boxes! Thus it looks as if the rules of quantum reasoning have given rise to a contradiction.

However, these rules, as summarized in Ch. 16, require that both the initial data and the conclusions be embedded in a *single framework*, whereas we have employed two different consistent families, $\mathcal{A}$ and $\mathcal{B}$. In addition, in order to reach a contradiction we used the assertion that A and B are mutually exclusive, and this requires a third framework $\mathcal{C}$, since $\mathcal{A}$ does not include B and $\mathcal{B}$ does not include A at t_1. If the frameworks $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ were compatible with each other, as is always the case in classical physics, there would be no problem, for the inferences carried out in the separate frameworks would be equally valid in the common refinement. But, as we shall show, these frameworks are mutually incompatible, despite the fact that the history projectors commute with one another.

Any common refinement of $\mathcal{A}$ and $\mathcal{B}$ would have to contain, among other things, the first history in (22.29) and the first one in (22.31):

$$D \odot A \odot F, \quad D \odot B \odot F. \tag{22.33}$$

The product of these two history projectors is zero, since $AB = 0$, but the chain operators are *not* orthogonal to each other. If one works out the chain kets one finds that they are both equal to a non-zero constant times $|F\rangle$. Thus having the two histories in (22.33) in the same family will violate the consistency conditions. A convenient choice for $\mathcal{C}$ is the family whose support is the three histories

$$D \odot \begin{cases} A \odot I, \\ B \odot I, \\ C \odot I. \end{cases} \tag{22.34}$$

Note that this is, in effect, a family of histories defined at only two times, t_0 and t_1, as I provides no information about what is going on at t_2, and for this reason it is automatically consistent, Sec. 11.3. It is incompatible with $\mathcal{A}$ because a common refinement would have to include the two histories

$$D \odot A \odot F, \quad D \odot B \odot I, \tag{22.35}$$

whose projectors are orthogonal, but whose chain kets are not, and it is likewise incompatible with $\mathcal{B}$.

Thus the paradox arises because of reasoning in a way which violates the single framework rule, and in this respect it resembles the two-spin paradox of Sec. 22.3. An important difference is that the incompatibility between frameworks in the case of two spins results from the fact that some of the nine operators in (22.14) do not commute with each other, whereas in the three-box paradox the projectors for the histories commute with each other, and incompatibility arises because the consistency conditions are not fulfilled in a common refinement.

Rewording the paradox in a slightly different way may assist in understanding why some types of inference which seem quite straightforward in terms of ordinary reasoning are not valid in quantum theory. Let us suppose that we have used the family $\mathcal{A}$ together with the initial data of D at t_0 and F at t_2 to reach the conclusion that at time t_1 A is true and $\tilde{A} = I - A$ is false. Since $\tilde{A} = B + C$, it seems natural to conclude that both B and C are false, contradicting the result (from framework $\mathcal{B}$) that B is true. The step from the falsity of $\tilde{A}$ to the falsity of B as a consequence of $\tilde{A} = B + C$ would be justified in classical mechanics by the following rule: If $P = Q + R$ is an indicator which is the sum of two other indicators, and P is false, meaning $P(\gamma) = 0$ for the phase point γ describing the physical system, then $Q(\gamma) = 0$ and $R(\gamma) = 0$, so both Q and R are false. For example, if the energy of a system is not in the range between 10 and 20 J, then it is not between 10 and 15 J, nor is it between 15 and 20 J.

The corresponding rule in quantum physics states that if the projector P is the sum of two projectors Q and R, and P is known to be false, then *if Q and R are part of the Boolean algebra of properties entering into the logical discussion*, both Q and R are false. The words in italics apply equally to the case of classical reasoning, but they are usually ignored, because if Q is not among the list of properties available for discussion, it can always be added, and $R = P - Q$ added at the same time to ensure that the properties form a Boolean algebra. In classical physics there is never any problem with adding a property which has not previously come up in the discussion, and therefore the rule in italics can safely be relegated to the dusty books on formal logic which scientists put off reading until after they retire. However, in quantum theory it is by no means the case that Q (and therefore $R = P - Q$) can always be added to the list of properties or events under discussion, and this is why the words in italics are extremely important. If by using the family $\mathcal{A}$ we have come to the conclusion that $\tilde{A} = B + C$ is false, and, as is in fact the case, B at t_1 *cannot* be added to this family while maintaining consistency, then B has to be regarded as meaningless from the point of view of the discussion based upon $\mathcal{A}$, and something which is meaningless cannot be either true or false.

One can also think about it as follows. The physicist who first uses the initial data and framework $\mathcal{A}$ to conclude that A was true at t_1, and then inserts B at t_1 into the discussion has, in effect, changed the framework to something other than $\mathcal{A}$. In classical physics such a change in framework causes no problems, and it certainly does not alter the correctness of a conclusion reached earlier in a framework which made no mention of B. But in the quantum case, adding B means that something else must be changed in order to ensure that one still has a consistent framework. Since A occurs at the end of the previous step of the argument and is thus still at the center of his attention, the physicist who introduces B is unconsciously (which is what makes the move so dangerous!) shifting to a framework, such as (22.34), in which either D at t_0 or F at t_2 has been forgotten. But as the new framework does not include the initial data, it is no longer possible to derive the truth of A. Hence adding B to the discussion in this manner is, relative to the truth of A, rather like sawing off the branch on which one is seated, and the whole argument comes crashing to the ground.

22.6 Truth Functionals for Histories

The notion of a truth functional can be applied to histories as well as to properties of a quantum system at a single time, and makes perfectly good sense as long as one considers a *single* framework or consistent family, based upon a sample space consisting of some decomposition of the history identity into elementary histories, as discussed in Sec. 8.5. Given this framework, one and only one of its elementary histories will actually occur, or be true, for a single quantum system during a particular time interval or run. A truth functional is then a function which assigns 1 (true) to a particular elementary history, 0 (false) to the other elementary histories, and 1 or 0 to other members of the Boolean algebra of histories using a formula which is the obvious analog of (22.21). The number of distinct truth functionals will typically be less than the number of elementary histories, since one need not count histories with zero weight—they are dynamically impossible, so they never occur—and certain elementary histories will be excluded by the initial data, such as an initial state.

A universal truth functional θ_u for histories can be defined in a manner analogous to a universal truth functional for properties, Sec. 22.4. We assume that θ_u assigns a value, 1 or 0, to every projector representing a history which is not intrinsically inconsistent (Sec. 11.8), i.e., any history which is a member of at least one consistent family, and that this assignment satisfies the first two conditions of (22.24) and the third condition whenever it makes sense. That is, (22.25) should hold when P and Q are two histories belonging to the same consistent family (which implies, among other things, that $PQ = QP$). For purposes of the following discussion it will be convenient to denote by $\mathcal{T}$ the collection of all true histories, the histories to which θ_u assigns the value 1. Given that θ_u satisfies these conditions, it is not hard to see that when it is restricted to a particular consistent family or framework $\mathcal{F}$, i.e., regarded as a function on the histories belonging to this family, it will coincide with one of the "ordinary" truth functionals for this family, and therefore $\mathcal{T} \cap \mathcal{F}$, the subset of all true histories belonging to $\mathcal{F}$, will consist of one elementary history and all compound histories which contain this particular elementary history. In particular, θ_u can never assign the value 1 to two distinct elementary histories belonging to the same framework.

Since a decomposition of the identity at a single time is an example, albeit a rather trivial one, of a consistent family of one-time "histories", it follows that there can be no truly universal truth functional for histories of a quantum system whose Hilbert space is of dimension three or more. Nonetheless, it interesting to see how the three-box paradox of the previous section provides an explicit example, with non-trivial histories, of a circumstance in which there is no universal truth functional. Imagine it as an experiment which is repeated many times, always starting with the same initial state D. The universal truth functional and the corresponding list $\mathcal{T}$ of true histories will vary from one run to the next, since different histories will occur in different runs. Think of a run (it will occur with a probability of 1/9) in which the final state is F, so the history $D \odot I \odot F$ is true, and therefore an element of $\mathcal{T}$. What other histories belong to $\mathcal{T}$, and are thus assigned the value 1 by the universal truth functional θ_u?

Consider the consistent family $\mathcal{A}$ whose sample space is shown in (22.29), aside from histories of zero weight to which θ_u will always assign the value 0. One and only one of these histories must be true, so it is the history $D \odot A \odot F$, as the other two terminate in $\tilde{F}$. From this we can conclude, using the counterpart of (22.21), that $D \odot A \odot I$, which belongs to the Boolean algebra of $\mathcal{A}$, is true, a member of $\mathcal{T}$. Following the same line of reasoning for the consistent family $\mathcal{B}$, we conclude that $D \odot B \odot F$ and $D \odot B \odot I$ are elements of $\mathcal{T}$. But now consider the consistent family $\mathcal{C}$ with sample space (22.34). One and only one of these three elementary histories can belong to $\mathcal{T}$, and this contradicts the conclusion we reached previously using $\mathcal{A}$ and $\mathcal{B}$, that *both* $D \odot A \odot I$ *and* $D \odot B \odot I$ belong to $\mathcal{T}$.

Our analysis does not by itself rule out the possibility of a universal truth functional which assigns the value 0 to the history $D \odot I \odot F$, and could be used in a run in which $D \odot I \odot \tilde{F}$ occurs. But it shows that the concept can, at best, be of rather limited utility in the quantum domain, despite the fact that it works without any difficulty in classical physics. Note that quantum truth functionals form a perfectly valid procedure for analyzing histories (and properties at a single time) as long as one restricts one's attention to a *single framework,* a single consistent family. With this restriction, quantum truth as it is embodied in a truth functional behaves in much the same way as classical truth. It is only when one tries to extend this concept of truth to something which applies simultaneously to different incompatible frameworks that problems arise.